Instructor's Manual to Accompany

FUNDAMENTAL METHODS OF MATHEMATICAL ECONOMICS

Fourth Edition

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Contents

CHAPTER 2

Exercise 2.3

1. (a) $\{x \mid x > 34\}$ (b) $\{x \mid 8 < x < 65\}$

2. True statements: (a), (d), (f), (g), and (h)

3. (a) {2,4,6,7} (b) {2,4,6} (c) {2,6}
   (d) {2} (e) {2} (f) {2,4,6}

4. All are valid.

5. First part: $A \cup (B \cap C) = \{4, 5, 6\} \cup \{3, 6\} = \{3, 4, 5, 6\}$; and $(A \cup B) \cap (A \cup C) = \{3, 4, 5, 6, 7\} \cap \{2, 3, 4, 5, 6\} = \{3, 4, 5, 6\}$ too.
   Second part: $A \cap (B \cup C) = \{4, 5, 6\} \cap \{2, 3, 4, 6, 7\} = \{4, 6\}$; and $(A \cap B) \cup (A \cap C) = \{4, 6\} \cup \{6\} = \{4, 6\}$ too.

6. N/A

7. $\emptyset$, {5}, {6}, {7}, {5, 6}, {5, 7}, {6, 7}, {5, 6, 7}

8. There are $2^4 = 16$ subsets: $\emptyset$, {a}, {b}, {c}, {d}, {a,b}, {a,c}, {a,d}, {b,c}, {b,d}, {c,d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}, and {a,b,c,d}.

9. The complement of U is $\tilde{U} = \{x \mid x \notin U\}$. Here the notation of "not in U" is expressed via the $\notin$ symbol which relates an *element* (x) to a *set* (U). In contrast, when we say "$\emptyset$ is a subset of U," the notion of "in U" is expressed via the $\subset$ symbol which relates a subset($\emptyset$) to a set (U). Hence, we have two different contexts, and there exists no paradox at all.

Exercise 2.4

1. (a) {(3,a), (3,b), (6,a), (6,b) (9,a), (9,b)}
   (b) {(a,m), (a,n), (b,m), (b,n)}
   (c) { (m,3), (m,6), (m,9), (n,3), (n,6), (n,9)}

2. {(3,a,m), (3,a,n), (3,b,m), (3,b,n), (6,a,m), (6,a,n), (6,b,m), (6,b,n), (9,a,m), (9,a,n), (9,b,m), (9,b,n),}

3. No. When $S_1 = S_2$.

4. Only (d) represents a function.

5. Range $= \{y \mid 8 \leq y \leq 32\}$

6. The range is the set of all nonpositive numbers.

7. (a) No. (b) Yes.

8. For each level of output, we should discard all the inefficient cost figures, and take the lowest cost figure as the total cost for that output level. This would establish the uniqueness as required by the definition of a function.

Exercise 2.5

1. N/a

2. Eqs. (a) and (b) differ in the sign of the coefficient of x; a positive (negative) sign means an upward (downward) slope.
   Eqs. (a) and (c) differ in the constant terms; a larger constant means a higher vertical intercept.

3. A negative coefficient (say, -1) for the x^2 term is associated with a hill. as the value of x is steadily increased or reduced, the $-x^2$ term will exert a more dominant influence in determining the value of y. Being negative, this term serves to pull down the y values at the two extreme ends of the curve.

4. If negative values can occur there will appear in quadrant III a curve which is the mirror image of the one in quadrant I.

5. (a) x^{19} (b) x^{a+b+c} (c) $(xyz)^3$

6. (a) x^6 (b) $x^{1/6}$

7. By Rules VI and V, we can successively write $x^{m/n} = (x^m)^{1/n} = \sqrt[n]{x^m}$; by the same two rules, we also have $x^{m/n} = (x^{1/n})^m = (\sqrt[n]{x})^m$

8. Rule VI:
$$(x^m)^n = \underbrace{x^m \times x^m \times ... \times x^m}_{\text{n terms}} = \underbrace{x \times x \times ... \times x}_{mn \text{ terms}} = x^{mn}$$

Rule VII:

$$\begin{aligned} x^m \times y^m &= \underbrace{x \times x \times ... \times x}_{m \text{ terms}} \times \underbrace{y \times y \ldots \times y}_{m \text{ terms}} \\ &= \underbrace{(xy) \times (xy) \times \ldots \times (xy)}_{m \text{ terms}} = (xy)^m \end{aligned}$$

CHAPTER 3

Exercise 3.2

1. (a) By substitution, we get $21 - 3P = -4 + 8P$ or $11P = 25$. Thus $P^* = 2\frac{3}{11}$. Substituting P^* into the second equation or the third equation, we find $Q^* = 14\frac{2}{11}$.

   (b) With $a = 21$, $b = 3, c = 4, d = 8$, the formula yields

$$P^* = \frac{25}{11} = 2\frac{3}{11} \quad Q^* = \frac{156}{11} = 14\frac{2}{11}$$

2.

   (a)

$$P^* = \frac{61}{9} = 6\frac{7}{9} \quad Q^* = \frac{276}{9} = 30\frac{2}{3}$$

   (b)

$$P^* = \frac{36}{7} = 5\frac{1}{7} \quad Q^* = \frac{138}{7} = 19\frac{5}{7}$$

3. N/A

4. If $b+d = 0$ then P^* and Q^* in (3.4) and (3.5) would involve division by zero, which is undefined.

5. If $b + d = 0$ then $d = -b$ and the demand and supply curves would have the same slope (though different vertical intercepts). The two curves would be parallel, with no equilibrium intersection point in Fig. 3.1

Exercise 3.3

1. (a) $x_1^* = 5$; $x_2^* = 3$ (b) $x_1^* = 4$; $x_2^* = -2$

2. (a) $x_1^* = 5$; $x_2^* = 3$ (b) $x_1^* = 4$; $x_2^* = -2$

3.

   (a) $(x - 6)(x + 1)(x - 3) = 0$, or $x^3 - 8x^2 + 9x + 18 = 0$

   (b) $(x - 1)(x - 2)(x - 3)(x - 5) = 0$, or $x^4 - 11x^3 + 41x^2 - 61x + 30 = 0$

4. By Theorem III, we find:

   (a) Yes. (b) No. (c) Yes.

5.

(a) By Theorem I, any integer root must be a divisor of 6; thus there are six candidates: ± 1, ± 2, and ± 3. Among these, -1, $\frac{1}{2}$, and $-\frac{1}{4}$

(b) By Theorem II, any rational root r/s must have r being a divisor of -1 and s being a divisor of 8. The r set is $\{1, -1\}$, and the s set is $\{1, -1, 2, -2, 4, -4, 8, -8\}$; these give us eight root candidates: $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$, and $\pm\frac{1}{8}$. Among these, -1, 2, and 3 satisfy the equation, and they constitute the three roots.

(c) To get rid of the fractional coefficients, we multiply every term by 8. The resulting equation is the same as the one in (b) above.

(d) To get rid of the fractional coefficients, we multiply every term by 4 to obtain

$$4x^4 - 24x^3 + 31x^2 - 6x - 8 = 0$$

By Theorem II, any rational root r/s must have r being a divisor of -8 and s being a divisor of 4. The r set is $\{\pm 1, \pm 2, \pm 4, \pm 8\}$, and the s set is $\{\pm 1, \pm 2, \pm 4\}$; these give us the root candidates $\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2, \pm 4, \pm 8$. Among these, $\frac{1}{2}$, $-\frac{1}{2}$, 2, and 4 constitute the four roots.

6.

(a) The model reduces to $P^2 + 6P - 7 = 0$. By the quadratic formula, we have $P_1^* = 1$ and $P_2^* = -7$, but only the first root is acceptable. Substituting that root into the second or the third equation, we find $Q^* = 2$.

(b) The model reduces to $2P^2 - 10 = 0$ or $P^2 = 5$ with the two roots $P_1^* = \sqrt{5}$ and $P_2^* = -\sqrt{5}$. Only the first root is admissible, and it yields $Q^* = 3$.

7. Equation (3.7) is the equilibrium stated in the form of "the excess supply be zero."

Exercise 3.4

1. N/A

2.

$$P_1^* = \frac{(a_2 - b_2)(\alpha_0 - \beta_0) - (a_0 - b_0)(\alpha_2 - \beta_2)}{(a_1 - b_1)(\alpha_2 - \beta_2) - (a_2 - b_2)(\alpha_1 - \beta_1)}$$

$$P_2^* = \frac{(a_0 - b_0)(\alpha_1 - \beta_1) - (a_1 - b_1)(\alpha_0 - \beta_0)}{(a_1 - b_1)(\alpha_2 - \beta_2) - (a_2 - b_2)(\alpha_1 - \beta_1)}$$

3. Since we have

$c_0 = 18 + 2 = 20 \quad c_1 = -3 - 4 = -7 \quad c_2 = 1$

$\gamma_0 = 12 + 2 = 14 \quad \gamma_1 = 1 \quad \gamma_2 = -2 - 3 = -5$

it follows that

$$P_1^* = \tfrac{14+100}{35-1} = \tfrac{57}{17} = 3\tfrac{6}{17} \quad \text{and} \quad P_2^* = \tfrac{20+98}{35-1} = \tfrac{59}{17} = 3\tfrac{8}{17}$$

Substitution into the given demand or supply function yields

$$Q_1^* = \tfrac{194}{17} = 11\tfrac{7}{17} \quad \text{and} \quad Q_2^* = \tfrac{143}{17} = 8\tfrac{7}{17}$$

Exercise 3.5

1.

(a) Three variables are endogenous: Y, C, and T.

(b) By substituting the third equation into the second and then the second into the first, we obtain

$$Y = a - bd + b(1-t)Y + I_0 + G_0$$

or

$$[1 - b(1-t)]Y = a - bd + I_0 + G_0$$

Thus

$$Y^* = \frac{a - bd + I_0 + G_0}{1 - b(1-t)}$$

Then it follows that the equilibrium values of the other two endogenous variables are

$$T^* = d + tY^* = \frac{d(1-b) + t(a + I_0 + G_0)}{1 - b(1-t)}$$

and

$$C^* = Y^* - I_0 - G_0 = \frac{a - bd + b(1-t)(I_0 + G_0)}{1 - b(1-t)}$$

2.

(a) The endogenous variables are Y, C, and G.

(b) $g = G/Y$ = proportion of national income spent as government expenditure.

(c) Substituting the last two equations into the first, we get

$$Y = a + b(Y - T_0) + I_0 + gY$$

Thus

$$Y^* = \frac{a - bT_0 + I_0}{1 - b - g}$$

(d) The restriction $b + g \neq 1$ is needed to avoid division by zero.

3. Upon substitution, the first equation can be reduced to the form

$$Y - 6Y^{1/2} - 55 = 0$$

or

$$w^2 - 6w - 55 = 0 \qquad (\text{where } w = Y^{1/2})$$

The latter is a quadratic equation, with roots

$$w_1^*,\ w_2^* = \left[\frac{1}{2}6 \pm (36 + 220)^{1/2}\right] = 11,\ -5$$

From the first root, we can get

$$Y^* = w_1^{*2} = 121 \quad \text{and} \quad C^* = 25 + 6(11) = 91$$

On the other hand, the second root is inadmissible because it leads to a negative value for C:

$$C^* = 25 + 6(-5) = -5$$

CHAPTER 4

Exercise 4.1

1.

$$\begin{aligned} Q_d \quad -Q_s \qquad\qquad &= 0 \\ Q_d \qquad\qquad +bP \quad &= a \\ Q_s \quad -dP \quad &= -c \end{aligned}$$

Coefficient Matrix:
$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & b \\ 0 & 1 & -d \end{bmatrix}$$

Vector of Constants:
$$\begin{bmatrix} 0 \\ a \\ -c \end{bmatrix}$$

2.

$$\begin{aligned}
Q_{d1} \quad -Q_{s1} \qquad\qquad\qquad\qquad\qquad &= 0 \\
Q_{d1} \qquad\qquad\qquad -a_1P_1 \quad -a_2P_2 \quad &= a_0 \\
Q_{s1} \qquad\qquad -b_1P_1 \quad -b_2P_2 \quad &= b_0 \\
Q_{d2} \quad -Q_{s2} \qquad\qquad\qquad &= 0 \\
Q_{d2} \qquad -\alpha_1P_1 \quad -\alpha_2P_2 \quad &= \alpha_0 \\
Q_{s2} \quad -\beta_1P_1 \quad -\beta_2P_2 \quad &= \beta_0
\end{aligned}$$

Coefficient matrix:
$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -a_1 & -a_2 \\ 0 & 1 & 0 & 0 & -b_1 & -b_2 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\alpha_1 & -\alpha_2 \\ 0 & 0 & 0 & 1 & -\beta_1 & -\beta_2 \end{bmatrix}$$

Variable vector:
$$\begin{bmatrix} Q_{d1} \\ Q_{s1} \\ Q_{d2} \\ Q_{s2} \\ P_1 \\ P_2 \end{bmatrix}$$

Constant vector:
$$\begin{bmatrix} 0 \\ a_0 \\ b_0 \\ 0 \\ \alpha_0 \\ \beta_0 \end{bmatrix}$$

3. No, because the equation system is nonlinear

4.

$$\begin{aligned} Y - C &= I_0 + G_0 \\ -bY + C &= a \end{aligned}$$

The coefficient matrix and constant vector are

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} \qquad \begin{bmatrix} I_0 + G_0 \\ a \end{bmatrix}$$

5. First expand the multiplicative expression $(b(Y-T)$ into the additive expression $bY - bT$ so that bY and $-bT$ can be placed in separate columns. Then we can write the system as

$$\begin{array}{llll} Y & & -C & = I_0 + G_0 \\ -bY & +bT & +C & = a \\ -tY & +T & & = d \end{array}$$

Exercise 4.2

1. (a) $\begin{bmatrix} 7 & 3 \\ 9 & 7 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 \\ 0 & -8 \end{bmatrix}$ (c) $\begin{bmatrix} 21 & -3 \\ 18 & 27 \end{bmatrix}$ (d) $\begin{bmatrix} 16 & 22 \\ 24 & -6 \end{bmatrix}$

2.

   (a) Yes $AB = \begin{bmatrix} 28 & 64 \\ 6 & 0 \\ 13 & 8 \end{bmatrix}$. No, not conformable.

   (b) Both are defined, but $BC = \begin{bmatrix} 14 & 4 \\ 69 & 30 \end{bmatrix} \neq CB = \begin{bmatrix} 20 & 16 \\ 21 & 24 \end{bmatrix}$

3. Yes. $BA = \begin{bmatrix} -\frac{1}{5} + \frac{12}{10} & 0 & -\frac{3}{5} + \frac{6}{10} \\ -3 + \frac{1}{5} + \frac{28}{10} & 1 & -2 + \frac{3}{5} + \frac{14}{10} \\ \frac{2}{5} - \frac{4}{10} & 0 & \frac{6}{5} - \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
   Thus we happen to have $AB = BA$ in this particular case.

4. (a) $\underset{(3\times2)}{\begin{bmatrix} 0 & 2 \\ 36 & 20 \\ 16 & 3 \end{bmatrix}}$ (b) $\underset{(2\times2)}{\begin{bmatrix} 49 & 3 \\ 4 & 3 \end{bmatrix}}$ (c) $\underset{(2\times1)}{\begin{bmatrix} 3x + 5y \\ 4x + 2y - 7z \end{bmatrix}}$ (d) $\underset{(1\times2)}{\begin{bmatrix} 7a + c & 2b + 4c \end{bmatrix}}$

5. Yes. Yes. Yes. Yes.

6.

   (a) $x_2 + x_3 + x_4 + x_5$

   (b) $a_5 + a_6x_6 + a_7x_7 + a_8x_8$

   (c) $b(x_1 + x_2 + x_3 + x_4)$

   (d) $a_1x^0 + a_2x^1 + \cdots + a_nx^{n-1} = a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1}$

(e) $x^2 + (x+1)^2 + (x+2)^2 + (x+3)^2$

7. (a) $\sum_{i=1}^{3} ix_i(x_i - 1)$ (b) $\sum_{i=2}^{4} a_i(x_{i+1} + i)$ (c) $\sum_{i=1}^{n} \frac{1}{x^i}$ (d) $\sum_{i=0}^{n} \frac{1}{x^i}$

8.

(a) $\left(\sum_{i=1}^{n} x_i\right) + x_{n+1} = x_0 + x_1 + \cdots + x_n + x_{n+1} = \sum_{i=1}^{n+1} x_i$

(b)

$$\begin{aligned} \sum_{j=1}^{n} ab_j y_j &= ab_1 y_1 + ab_2 y_2 + \cdots + ab_n y_n \\ &= a(b_1 y_1 + b_2 y_2 + \cdots + b_n y_n) = a \sum_{j=1}^{n} b_j y_j \end{aligned}$$

(c)

$$\begin{aligned} \sum_{j=1}^{n} (x_j + y_j) &= (x_1 + y_1) + (x_2 + y_2) + \cdots + (x_n + y_n) \\ &= (x_1 + x_2 + \cdots + x_n) + (y_1 + y_2 + \cdots + y_n) \\ &= \sum_{j=1}^{n} x_j + \sum_{j=1}^{n} y_j \end{aligned}$$

Exercise 4.3

1.

(a) $uv' = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 15 & 5 & -5 \\ 3 & 1 & -1 \\ 9 & 3 & -3 \end{bmatrix}$

(b) $uw' = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 7 & 8-1 \end{bmatrix} = \begin{bmatrix} 35 & 25 & 40 \\ 7 & 5 & 8 \\ 21 & 15 & 24 \end{bmatrix}$

(c) $xx' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{bmatrix}$

(d) $v'u = \begin{bmatrix} 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = [15 + 1 - 3] = [44] = 44$

(e) $u'v = \begin{bmatrix} 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = [15 + 1 - 3] = 13$

(f) $w'x = \begin{bmatrix} 7 & 5 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [7x_1 + 5x_2 + 8x_3] = 7x_1 + 5x_2 + 8x_3$

(g) $u'u = \begin{bmatrix} 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = [25 + 1 + 9] = [35] = 35$

(h) $x'x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left[x_1^2 + x_2^2 + x_3^2\right] = \sum_{i=1}^{3} x_i^2$

2.

(a) All are defined except $w'x$ and $x'y'$.

(b) $xy' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 \\ x_2y_1 & x_2y_2 \end{bmatrix}$

$xy' = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = y_1^2 + y_2^2$

$zz' = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \end{bmatrix} = \begin{bmatrix} z_1^2 & z_1z_2 \\ z_2z_1 & z_2^2 \end{bmatrix}$

$yw' = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 16 \end{bmatrix} = \begin{bmatrix} 3y_1 & 2y_1 & 16y_1 \\ 3y_2 & 2y_2 & 16y_2 \end{bmatrix}$

$x \cdot y = x_1y_1 + x_2y_2$

3.

(a) $\sum_{i=1}^{n} P_iQ_i$

(b) Let P and Q be the column vectors or prices and quantities, respectively. Then the total revenue is $P \cdot Q$ or $P'Q$ or $Q'P$.

4.

(a) $w_1'w_2 = 11$ (acute angle, Fig. 4.2c)

(b) $w_1'w_2 = -11$ (obtuse angle, Fig. 4.2d)

(c) $w_1'w_2 = -13$ (obtuse angle, Fig. 4.2b)

(d) $w_1'w_2 = 0$ (right angle, Fig. 4.3)

(e) $w_1'w_2 = 5$ (acute angle, Fig. 4.3)

5. (a) $2v = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ (b) $u + v = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ (c) $u - v = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$

(d) v$-u = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ (e) $2u + 3v = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$ (f) $4u - 2v = \begin{bmatrix} 20 \\ -2 \end{bmatrix}$

6. (a) $4e_1 + 7e_2$ (b) $25e_1 - 2e_2 + e_3$

(c) $-e_1 + 6e_2 + 9e_3$ (d) $2e_1 + 8e_3$

7.

(a) $d = \sqrt{(3-0)^2 + (2+1)^2 + (8-5)^2} = \sqrt{27}$

(b) $d = \sqrt{(9-2)^2 + 0 + (4+4)^2} = \sqrt{113}$

8. When u, v, and w all lie on a single straight line.

9. Let the vector v have the elements $(a_1, \ldots, a_n)$. The point of origin has the elements $(0, \ldots, 0)$. Hence:

(a) $$d(0,v) = d(v,0) = \sqrt{(a_1 - 0)^2 + \ldots + (a_n - 0)^2} = \sqrt{a_1^2 + \ldots + a_n^2}$$

(b) $d(v,0) = (v'v)^{1/2}$ [See Example 3 in this section]

(c) $d(v,0) = (v \cdot v)^{1/2}$

Exercise 4.4

1.

(a) $(A + B) + C = A + (B + C) = \begin{bmatrix} 5 & 17 \\ 11 & 17 \end{bmatrix}$

(b) $(A+B)+C = A+(B+C) = \begin{bmatrix} -1 & 9 \\ 9 & -1 \end{bmatrix}$

2. No. It should be $A - B = -B + A$

3. $(AB)C = A(BC) = \begin{bmatrix} 250 & 68 \\ 75 & 55 \end{bmatrix}$

(a) $$\begin{aligned} k(A+B) &= k[a_{ij}+b_{ij}] = [ka_{ij}+kb_{ij}] = [ka_{ij}] + [kb_{ij}] \\ &= k[a_{ij}] + k\,[b_{ij}] = kA + kB \end{aligned}$$

(b) $$\begin{aligned} (g+k)A &= (g+k)[a_{ij}] = [(g+k)a_{ij}] = [ga_{ij}+ka_{ij}] \\ &= [ga_{ij}] + [ka_{ij}] = g\,[a_{ij}] + k\,[a_{ij}] = gA + kA \end{aligned}$$

4.

(a)

$$\begin{aligned} AB &= \begin{bmatrix} (12\times 3)+(14\times 0) & (12\times 9)+(14\times 2) \\ (20\times 3)+(5\times 0) & (20\times 9)+(5\times 2) \end{bmatrix} \\ &= \begin{bmatrix} 36 & 136 \\ 60 & 190 \end{bmatrix} \end{aligned}$$

(b)

$$\begin{aligned} AB &= \begin{bmatrix} (4\times 3)+(7\times 2) & (4\times 8)+(7\times 6) & (4\times 5)+(7\times 7) \\ (9\times 3)+(1\times 2) & (9\times 8)+(1\times 6) & (9\times 5)+(1\times 7) \end{bmatrix} \\ &= \begin{bmatrix} 26 & 74 & 69 \\ 29 & 78 & 52 \end{bmatrix} \end{aligned}$$

(c)

$$\begin{aligned} AB &= \begin{bmatrix} (7\times 12)+(11\times 3) & (7\times 4)+(11\times 6) & (7\times 5)+(11\times 1) \\ (2\times 12)+(9\times 3) & (2\times 4)+(9\times 6) & (2\times 5)+(9\times 1) \\ (10\times 12)+(6\times 3) & (10\times 4)+(6\times 6) & (10\times 5)+(6\times 1) \end{bmatrix} \\ &= \begin{bmatrix} 117 & 94 & 46 \\ 51 & 62 & 19 \\ 138 & 76 & 56 \end{bmatrix} = C \end{aligned}$$

(d)

$$\begin{aligned} AB &= \begin{bmatrix} (6\times 10)+(2\times 11)+(5\times 2) & (6\times 1)+(2\times 3)+(5\times 9) \\ (7\times 10)+(9\times 11)+(4\times 2) & (7\times 1)+(9\times 3)+(4\times 9) \end{bmatrix} \\ &= \begin{bmatrix} 92 & 57 \\ 177 & 70 \end{bmatrix} \end{aligned}$$

(e)

i. $AB = \begin{bmatrix} -2\times 3 & -2\times 6 & -2\times -2 \\ 4\times 3 & 4\times 6 & 4\times -2 \\ 7\times 3 & 7\times 6 & 7\times -2 \end{bmatrix} = \begin{bmatrix} -6 & -12 & 4 \\ 12 & 24 & -8 \\ 21 & 42 & -14 \end{bmatrix}$

ii. $BA = [(3\times -2)+(6\times 4)+(-2\times 7)] = [4]$

5. $(A+B)(C+D) = (A+B)C + (A+B)D = AC + BC + AD + BD$

6. No, $x'Ax$ would then contain cross-product terms $a_{12}x_1x_2$ and $a_{21}x_1x_2$.

7. Unweighted sum of squares is used in the well-known method of least squares for fitting an equation to a set of data. Weighted sum of squares can be used, e.g., in comparing weather conditions of different resort areas by measuring the deviations from an ideal temperature and an ideal humidity.

Exercise 4.5

1.

(a) $AI_3 = \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix}$

(b) $I_2A = \begin{bmatrix} -1 & 5 & 7 \\ 0 & -2 & 4 \end{bmatrix}$

(c) $I_2x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(d) $x'I_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

2.

(a) $Ab = \begin{bmatrix} -9+30+0 \\ 0-12+0 \end{bmatrix} = \begin{bmatrix} 21 \\ -12 \end{bmatrix}$

(b) AIb gives the same result as in (a).

(c) $x'IA = \begin{bmatrix} -x_1 & 5x_1 - 2x_2 & 7x_1 + 4x_2 \end{bmatrix}$

(d) $x'A$ gives the same result as in (c)

3. (a) 5×3 (b) 2×6 (c) 2×1 (d) 2×5

4. The given diagonal matrix, when multiplied by itself, gives another diagonal matrix with the diagonal elements $a_{11}^2, a_{22}^2, \ldots, a_{nn}^2$. For idempotency, we must have $a_{ii}^2 = a_{ii}$ for every i. Hence each a_{ii} must be either 1, or 0. Since each a_{ii} can thus have two possible values, and since there are altogether n of these a_{ii}, we are able to construct a total of 2^n idempotent matrices of the diagonal type. Two examples would be I_n and 0_n.

Exercise 4.6

1. $A' = \begin{bmatrix} 0 & -1 \\ 4 & 3 \end{bmatrix}$ $B' = \begin{bmatrix} 3 & 0 \\ -8 & 1 \end{bmatrix}$ $C' = \begin{bmatrix} 1 & 6 \\ 0 & 1 \\ 9 & 1 \end{bmatrix}$

2. (a) $(A+B)' = A' + B' = \begin{bmatrix} 3 & -1 \\ -4 & 3 \end{bmatrix}$ (b) $(AC)' = C'A' = \begin{bmatrix} 24 & 17 \\ 4 & 3 \\ 4 & -6 \end{bmatrix}$

3. Let $D \equiv AB$. Then $(ABC)' \equiv (DC)' = C'D' = C'(AB)' = C'(B'A') = C'B'A'$

4. $DF = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, thus D and F are inverse of each other, Similarly,

$EG = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so E and G are inverses of each other.

5. Let $D \equiv AB$. Then $(ABC)^{-1} \equiv (DC)^{-1} = C^{-1}D^{-1} = C^{-1}(AB)^{-1} = C^{-1}(B^{-1}A^{-1}) = C^{-1}B^{-1}A^{-1}$

6.

(a) A and $X'X$ must be square, say $n \times n$; X only needs to be $n \times m$, where m is not necessarily equal to n.

(b) $AA = [I - X(X'X)^{-1}X'][I - X(X'X)^{-1}X']$

$= II - IX(X'X)^{-1}X' - X(X'X)^{-1}X'I + X(X'X)^{-1}X'X(X'X)^{-1}X'$

[see Exercise 4.4-6]

$= I - X(X'X)^{-1}X' - X(X'X)^{-1}X' + XI(X'X)^{-1}X'$ [by (4.8)]

$= I - X(X'X)^{-1}X'$

$= A$

Thus A satisfies the condition for idempotency.

Exercise 4.7

1. It is suggested that this particular problem could also be solved using a spreadsheet or other mathematical software. The student will be able to observe features of a Markov process more quickly without doing the repetitive calculations.

(a) The Markov transition matrix is $\begin{bmatrix} 0.9 & 0.1 \\ 0.7 & 0.3 \end{bmatrix}$

(b)

	Two periods	Three Periods	Five Periods	Ten Periods
Employed	1008	1042	1050	1050
Unemployed	192	158	150	150

(c) As the original Markov transition matrix is raised to successively greater powers the resulting matrix converges to

$$\underset{n\to\infty}{M^n} \Longrightarrow \begin{bmatrix} 0.875 & 0.125 \\ 0.875 & 0.125 \end{bmatrix}$$

which is the "steady state", giving us 1050 employed and 150 unemployed.

CHAPTER 5

Exercise 5.1

1. (a) (5.2) (b) (5.2) (c) (5.3) (d) (5.3) (e) (5.3)
   (f) (5.1) (g) (5.2)

2. (a) $p \Longrightarrow q$ (b) $p \Longrightarrow q$ (c) $p \Longleftrightarrow q$

3. (a) Yes (b) Yes (c) Yes (d) No; $v_2' = -2v_1'$

4. We get the same results as in the preceding problem.

   (a) Interchange row 2 and row 3 in A to get a matrix A_1. In A_1 keep row 1 as is, but add row 1 to row 2, to get A_2. In A_2, divide row 2 by 5. Then multiply the new row 2 by -3, and add the result to row 3. The resulting echelon matrix

$$A_3 = \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 8\frac{2}{5} \end{bmatrix}$$

   contains three nonzero-rows; hence $r(A) = 3$.

   (b) Interchange row 1 and row 3 in B to get a matrix B_1. In B_1, divide row 1 by 6. Then multiply the new row 1 by -3, and add the result to row 2, to get B_2. In B_2, multiply row 2 by 2, then add the new row 2 to row 3. The resulting echelon matrix

$$B_3 = \begin{bmatrix} 1 & \frac{1}{6} & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

   with two nonzero-rows in B_3; hence $r(B) = 2$. There is linear dependence in B: row 1 is equal to row 3 – 2(row 2). Matrix is singular.

   (c) Interchange row 2 and row 3 in C, to get matrix C_1. In C_1 divide row 1 to 7. Then multiply the new row 1 by -8, and add the result to row 2, to get C_2. In C_2, multiply row 2 by $-7/48$. Then multiply the new row 2 by -1 and add the result to row 3, to get

C_3. In C_3, multiply row 3 by 2/3, to get the echelon matrix

$$C_4 = \begin{bmatrix} 1 & \frac{6}{7} & \frac{3}{7} & \frac{3}{7} \\ 0 & 1 & \frac{1}{2} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{10}{9} \end{bmatrix}$$

There are three nonzero-rows in C_4; hence $r(C) = 3$. The question of nonsingularity is not relevant here because C is not square.

(d) interchange row 1 and row 2 in D, to get matrix D_1 (This step is optional, because we can just as well start by dividing the original row 1 by 2 to produce the desired unit element at the left end of the row. But the interchange of rows 1 and 2 gives us simpler numbers to work with). In D_1, multiply row 1 by -2, and add the result to row 2, to get D_2. Since the last two rows of D_2, are identical, linear dependence is obvious. To produce an echelon matrix, divide row 2 in D_2 by 5, and then add (-5) times the new row 2 to row 3. The resulting echelon matrix

$$D_3 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{9}{5} & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

contains two nonzero-rows; hence $r(D) = 2$. Again, the question nonsingularity is not relevant here.

5. The link is provided by the third elementary row operation. If, for instance, row 1 of a given matrix is equal to row 2 minus k times row 3 (showing a specific pattern of linear combination), then by adding (-1) times row 2 and k times row 3 to row 1, we can produce a zero-row. This process involves the third elementary row operation. the usefulness of the echelon matrix transformation lies in its systematic approach to force out zero-rows if they exist.

Exercise 5.2

1. (a) -6 (b) 0 (c) 0 (d) 157
   (e) $3abc - a^3 - b^3 - c^3$ (f) $8xy + 2x - 30$

2. $+$, $-$, $+$, $-$, $-$.

3. $|M_a| = \begin{vmatrix} a & f \\ h & i \end{vmatrix}$ $|M_b| = \begin{vmatrix} d & f \\ g & i \end{vmatrix}$ $|M_f| = \begin{vmatrix} a & b \\ g & h \end{vmatrix}$

   $|C_a| = |M_a|$ $|C_b| = -|M_b|$ $|C_f| = -|M_f|$

4. (a) 72 (b) −81

5. The cofactor of element 9 is $-\begin{vmatrix} 2 & 3 & 4 \\ 1 & 6 & 0 \\ 0 & -5 & 0 \end{vmatrix} = 20$

6. First find the minors

$$\begin{aligned} |M|_{31} &= \begin{vmatrix} 11 & 4 \\ 2 & 7 \end{vmatrix} = 69 \\ |M|_{32} &= \begin{vmatrix} 9 & 4 \\ 3 & 7 \end{vmatrix} = 51 \\ |M|_{33} &= \begin{vmatrix} 9 & 11 \\ 3 & 2 \end{vmatrix} = -15 \end{aligned}$$

   Step 4: Since a cofactor is simply the minor with a particular sign, according to $|C_{ij}| = (-1)^{i+j} |M_{ij}|$ we find:

$$\begin{aligned} |C_{31}| &= (-1)^4 |M_{31}| = 69 \\ |C_{32}| &= (-1)^5 |M_{32}| = -51 \\ |C_{33}| &= (-1)^6 |M_{33}| = -15 \end{aligned}$$

7. Expand second column

$$\begin{aligned} |A| &= a_{12} |C_{12}| + a_{22} |C_{22}| + a_{32} |C_{32}| \\ |A| &= (7)(-1)\begin{vmatrix} 2 & 6 \\ 9 & 12 \end{vmatrix} + (5)\begin{vmatrix} 15 & 9 \\ 9 & 12 \end{vmatrix} + 0 \\ |A| &= (7)(-30) + (5)(99) = 705 \end{aligned}$$

Exercise 5.3

1. N/A

2. Factoring out the k in each successive column (or row)–for a total of n columns (or rows)–will yield the indicated result.

3. (a) Property IV. (b) Property III (applied to both rows).

4. (a) Singular. (b) Singular. (c) Singular. (d) Nonsingular.

5. In (d), the rank is 3. In (a), (b) and (c), the rank is less than 3.

6. The set in (a) can because when the three vectors are combined into a matrix, its determinant does not vanish. But the set in (b) cannot.

7. A is nonsingular because $|A| = 1 - b \neq 0$.

   (a) To have a determinant, A has to be square.

   (b) Multiplying every element of an $n \times n$ determinant will increase the value of the determinant 2^n−fold. (See Problem 2 above)

   (c) Matrix A, unlike $|A|$, cannot "vanish." Also, an equation system, unlike a matrix, cannot be nonsingular or singular.

Exercise 5.4

1. They are $\sum_{i=1}^{4} a_{i3} |C_{i2}|$ and $\sum_{j=1}^{4} a_{2j} |C_{4j}|$, respectively.

2. Since $adjA = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$, We have $A^{-1} = \frac{AdjA}{|A|} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

   Similarly, we have $B^{-1} = -\frac{1}{2} \begin{bmatrix} 2 & 0 \\ -9 & -1 \end{bmatrix}$, $C^{-1} = -\frac{1}{24} \begin{bmatrix} -1 & -7 \\ -3 & 3 \end{bmatrix}$

3.

   (a) Interchange the two diagonal elements of A, multiply the two off-diagonal elements of A by −1.

   (b) Divide the $adjA$ by $|A|$.

4. $E^{-1} = \frac{1}{20}\begin{bmatrix} 3 & 2 & -3 \\ -7 & 2 & 7 \\ -6 & -4 & 26 \end{bmatrix}$, $F^{-1} = \frac{-1}{10}\begin{bmatrix} 0 & 2 & -3 \\ 10 & -6 & -1 \\ 0 & -4 & 1 \end{bmatrix}$,

   $G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $H^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

5. $A^{-1} = \frac{1}{|A|} \cdot AdjA = \frac{1}{98}\begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix} = \begin{bmatrix} \frac{13}{98} & \frac{1}{98} & \frac{16}{98} \\ \frac{11}{98} & \frac{31}{98} & \frac{6}{98} \\ -\frac{1}{14} & \frac{1}{14} & \frac{1}{7} \end{bmatrix}$

6.

   (a)

$$\begin{aligned} x &= A^{-1}d \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \frac{5}{14} & \frac{-3}{14} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}\begin{bmatrix} 28 \\ 42 \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{5}{14}\right)(28) + \left(\frac{-3}{14}\right)(42) \\ \left(-\frac{1}{7}\right)(28) + \left(\frac{2}{7}\right)(42) \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix} \end{aligned}$$

   (b)

$$\begin{aligned} x &= A^{-1}d \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} \frac{13}{98} & \frac{1}{98} & \frac{16}{98} \\ \frac{11}{98} & \frac{31}{98} & \frac{6}{98} \\ -\frac{1}{14} & \frac{1}{14} & \frac{1}{7} \end{bmatrix}\begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{104}{98} + & \frac{12}{98} + & \frac{80}{98} \\ \frac{88}{98} + & \frac{372}{98} + & \frac{30}{98} \\ -\frac{8}{14} + & \frac{12}{14} + & \frac{5}{7} \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \end{aligned}$$

7. Yes, Matrices G and H in problem 4 are examples.

Exercise 5.5

1.

   (a) $|A| = 7$, $|A_1| = 28$, $|A_2| = 21$, Thus $x_1^* = 4$, $x_2^* = 3$.

   (b) $|A| = -11$, $|A_1| = -33$, $|A_2| = 0$, Thus $x_1^* = 3$, $x_2^* = 0$.

   (c) $|A| = 15$, $|A_1| = 30$, $|A_2| = 15$, Thus $x_1^* = 2$, $x_2^* = 1$.

(d) $|A| = |A_1| = |A_2| = -78$, Thus $x_1^* = x_2^* = 1$.

2.

(a) $A^{-1} = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$ and $x^* = A^{-1}d = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

(b) $A^{-1} = \frac{-1}{11}\begin{bmatrix} -1 & -3 \\ -4 & -1 \end{bmatrix}$ and $x^* = A^{-1}d = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

(c) $A^{-1} = \frac{1}{15}\begin{bmatrix} 1 & 7 \\ -1 & 8 \end{bmatrix}$ and $x^* = A^{-1}d = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(d) $A^{-1} = \frac{-1}{78}\begin{bmatrix} -3 & -9 \\ -7 & 5 \end{bmatrix}$ and $x^* = A^{-1}d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3.

(a) $|A| = 38$, $|A_1| = 76$, $|A_2| = 0$, $|A_3| = 38$; thus $x_1^* = 2$, $x_2^* = 0$, $x_3^* = 1$.

(b) $|A| = 18$, $|A_1| = -18$, $|A_2| = 54$, $|A_3| = 126$; thus $x_1^* = -1$, $x_2^* = 3$, $x_3^* = 7$.

(c) $|A| = 17$, $|A_1| = 0$, $|A_2| = 51$, $|A_3| = 68$; thus $x^* = 0$, $y^* = 3$, $z^* = 4$.

(d) $|A| = 4$, $|A_1| = 2(b+c)$, $|A_2| = 2(a+c)$, $|A_3| = 2(a+b)$; thus $x^* = \frac{1}{2}(b+c)$, $y^* = \frac{1}{2}(a+c)$, $z^* = \frac{1}{2}(a+b)$.

4. After the indicated multiplication by the appropriate cofactors, the new equations will add up to the following equation:

$$\sum_{i=1}^{n} a_{i1}\,|C_{ij}|\,x_1 + \sum_{i=1}^{n} a_{i2}\,|C_{ij}|\,x_2 + \cdots + \sum_{i=1}^{n} a_{in}\,|C_{ij}|\,x_n = \sum_{i=1}^{n} d_i\,|C_{ij}|$$

When $j = 1$, the coefficient of x_1 becomes $|A|$, whereas the coefficients of the other variables all vanish; thus the last equation reduces to $|A|\,x_1 = \sum_{i=1}^{n} d_i\,|C_{i1}|$, leading to the result for x_1^* in (5.17). When $j = 2$, we similarly get the result for x_2^*.

Exercise 5.6

1. The system can be written as $\begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} I_0 + G_0 \\ a \\ d \end{bmatrix}$

(a) Since $A^{-1} = \frac{1}{1-b+bt}\begin{bmatrix} 1 & -1 & -b \\ b(1-t) & 1 & -b \\ t & t & 1-b \end{bmatrix}$, the solution is

$$\begin{bmatrix} Y^* \\ C^* \\ T^* \end{bmatrix} = A^{-1}d = \frac{1}{1-b+bt}\begin{bmatrix} I_0 + G_0 + a - bd \\ b(1-t)(I_0+G_0) + a - bd \\ t(I_0+G_0) + at + d(1-b) \end{bmatrix}$$

(b) $|A| = 1 - b + bt$

$|A_1| = I_0 + G_0 - bd + a$

$|A_2| = a - bd + b(1-t)(I_0 + G_0)$

$|A_3| = d(1-b) + t(a + I_0 + G_0)$

Thus

$$\begin{aligned} Y^* &= \frac{I_0 + G_0 + a - bd}{1-b+bt} \\ C^* &= \frac{a - bd + b(1-t)(I_0+G_0)}{1-b+bt} \\ T^* &= \frac{d(1-b) + t(I_0 + G_0 + a)}{1-b+bt} \end{aligned}$$

2. The system can be written as $\begin{bmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ -g & 0 & 1 \end{bmatrix}\begin{bmatrix} Y \\ C \\ G \end{bmatrix} = \begin{bmatrix} I_0 \\ a - bT_0 \\ 0 \end{bmatrix}$

(a) Since $A^{-1} = \frac{1}{1-b-g}\begin{bmatrix} 1 & 1 & 1 \\ b & 1-g & b \\ g & g & 1-b \end{bmatrix}$, the solution is

$$\begin{bmatrix} Y^* \\ C^* \\ G^* \end{bmatrix} = A^{-1}d = \frac{1}{1-b-g}\begin{bmatrix} I_0 + a - bT_0 \\ bI_0 + (1-g)(a - bT_0) \\ g(I_0 + a - bT_0) \end{bmatrix}$$

(b) $|A| = 1 - b - g$

$|A_1| = I_0 + a - bT_0$

$|A_2| = bI_0 + (1-g)(a - bT_0)$

$|A_3| = g(I_0 + a - bT_0)$

Thus

$$\begin{aligned} Y^* &= \frac{I_0 + a - bT_0}{1 - b - g} \\ C^* &= \frac{bI_0 + (1-g)(a - bT_0)}{1 - b - g} \\ G^* &= \frac{g(I_0 + a - bT_0)}{1 - b - g} \end{aligned}$$

3.

(a)

$$\begin{bmatrix} .3 & 100 \\ .25 & -200 \end{bmatrix} \begin{pmatrix} Y \\ R \end{pmatrix} = \begin{pmatrix} 252 \\ 176 \end{pmatrix}$$

(b) The inverse of A is

$$\begin{aligned} A^{-1} &= \frac{1}{|A|} \cdot Adj A = \frac{1}{-85} \begin{bmatrix} -200 & -.25 \\ -100 & .3 \end{bmatrix} \\ &= \begin{bmatrix} \frac{40}{17} & \frac{20}{17} \\ \frac{.05}{17} & \frac{-.06}{17} \end{bmatrix} \end{aligned}$$

Finally,

$$\begin{pmatrix} Y \\ R \end{pmatrix} = \begin{bmatrix} \frac{40}{17} & \frac{20}{17} \\ \frac{.05}{17} & \frac{-.06}{17} \end{bmatrix} \begin{bmatrix} 252 \\ 176 \end{bmatrix} = \begin{bmatrix} 800 \\ .12 \end{bmatrix}$$

Exercise 5.7

1. $\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \frac{1}{0.384} \begin{bmatrix} 0.66 & 0.30 & 0.24 \\ 0.34 & 0.62 & 0.24 \\ 0.21 & 0.27 & 0.60 \end{bmatrix} \begin{bmatrix} 30 \\ 15 \\ 10 \end{bmatrix} = \frac{1}{0.384} \begin{bmatrix} 26.70 \\ 21.90 \\ 16.35 \end{bmatrix} = \begin{bmatrix} 69.53 \\ 57.03 \\ 42.58 \end{bmatrix}$

2. $\sum_{j=1}^{3} a_{0j} x_j^* = 0.3(69.53) + 0.3(57.03) + 0.4(42.58) = \$55.00$ billion.

3.

(a) $A = \begin{bmatrix} 0.10 & 0.50 \\ 0.60 & 0 \end{bmatrix}$, $I - A = \begin{bmatrix} 0.90 & -0.50 \\ -0.60 & 1.00 \end{bmatrix}$. Thus the matrix equation is

$$\begin{bmatrix} 0.90 & -0.50 \\ -0.60 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$$

(b) The leading principle minors of the Leontief matrix are $|B_1| = 0.90 > 0$, $|B_1| = |I - A| = 0.60 > 0$, thus the Hawkins-Simon condition is satisfied.

(c) $x_1^* = \frac{2000}{0.60} = 3333\frac{1}{3}$ $\quad x_2^* = \frac{2400}{0.60} = 4000$

4.

(a) Element 0.33: 33c of commodity II is needed as input for producing \$1 of commodity I.

Element 0: Industry III does not use its own output as its input.

Element 200: The open sector demands 200 (billion dollars) of commodity II.

(b) Third-column sum = 0.46, meaning that 46c of non-primary inputs are used in producing \$1 of commodity III.

(c) No significant economic meaning.

(d) $$\begin{bmatrix} 0.95 & -0.25 & -0.34 \\ -0.33 & 0.90 & -0.12 \\ -0.19 & -0.38 & 1.00 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1800 \\ 200 \\ 900 \end{bmatrix}$$

(e) $|B_1| = 0.95 > 0 \qquad |B_2| = \begin{vmatrix} 0.95 & -0.25 \\ -0.33 & 0.90 \end{vmatrix} = 0.7725 > 0 \qquad |B_3| = |I - A| = 0.6227 > 0$

The Hawkins-Simon condition is satisfied.

5.

(a) 1st-order: $|B_{11}|$, $|B_{22}|$, $|B_{33}|$, $|B_{44}|$

2nd-order: $\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}, \begin{vmatrix} b_{11} & b_{13} \\ b_{31} & b_{33} \end{vmatrix}, \begin{vmatrix} b_{11} & b_{14} \\ b_{41} & b_{44} \end{vmatrix}$
$\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}, \begin{vmatrix} b_{22} & b_{24} \\ b_{42} & b_{44} \end{vmatrix}, \begin{vmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{vmatrix}$

3rd-order: $\begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix}, \begin{vmatrix} b_{11} & b_{12} & b_{14} \\ b_{21} & b_{22} & b_{24} \\ b_{41} & b_{42} & b_{44} \end{vmatrix}, \begin{vmatrix} b_{11} & b_{12} & b_{14} \\ b_{31} & b_{33} & b_{34} \\ b_{41} & b_{43} & b_{44} \end{vmatrix}, \begin{vmatrix} b_{22} & b_{23} & b_{24} \\ b_{32} & b_{33} & b_{34} \\ b_{42} & b_{43} & b_{44} \end{vmatrix}$

4th-order: same as $|B|$.

(b) The first three leading principal minors are the same as those in (5.28). the fourth one is simply $|B|$.

6. The last part of the Hawkins-Simon condition, $|B_n| > 0$, is equivalent to $|B| > 0$. Since $|B|$ is a nonsingular matrix, and $Bx = d$ has a unique solution $x^* = B^{-1}d$, not necessarily nonnegative.

CHAPTER 6

Exercise 6.2

1.

(a) $\frac{\Delta y}{\Delta x} = \frac{4(x+\Delta x)^2 + 9 - (4x^2+9)}{\Delta x} = 8x + 4\Delta x$

(b) $dy/dx = f'(x) = 8x$

(c) $f'(3) = 24$ and $f'(4) = 32$.

2.

(a) $\frac{\Delta y}{\Delta x} = 10x + 5\Delta x - 4$

(b) $dy/dx = 10x - 4$

(c) $f'(2) = 16 \qquad f'(3) = 26$

3.

(a) $\frac{\Delta y}{\Delta x} = 5$; a constant function.

(b) No; $dy/dx = 5$.

Exercise 6.4

1. Left-side limit = right-side limit = 15. Yes, the limit is 15.

2. The function can be rewritten as $q = (v^3 + 6v^2 + 12)/v = v^2 + 6v + 12$ $(v \neq 0)$. Thus

   (a) $\lim_{v \to 0} q = 12$ (b) $\lim_{v \to 2} q = 28$ (c) $\lim_{v \to a} q = a^2 + 6a + 12$

3. (a) 5 (b) 5

4. If we choose a very small neighborhood of the point $L + a_2$, we cannot find a neighborhood of N such that for every value of v in the N-neighborhood, q will be in the $(L + a_2)$-neighborhood.

Exercise 6.5

1.

   (a) Adding $-3x - 2$ to both sides, we get $-3 < 4x$. Multiplying both sides of the latter by $1/4$, we get the solution $-3/4 < x$.

   (b) The solution is $x < -9$.

   (c) The solution is $x < 1/2$

   (d) The solution is $-3/2 < x$.

2. The continued inequality is $8x - 3 < 0 < 8x$. Adding $-8x$ to all sides, and then multiplying by $-1/8$ (thereby reversing the sense of inequality), we get the solution $0 < x < 3/8$.

   (a) By (6.9), we can write $-6 < x + 1 < 6$. Subtracting 1 from all sides, we get $-7 < x < 5$ as the solution.

   (b) The solution is $2/3 < x < 2$.

   (c) The solution is $-4 \le x \le 1$.

Exercise 6.6

1.

   (a) $\lim\limits_{v\to 0} q = 7 - 0 + 0 = 7$

   (b) $\lim\limits_{v\to 3} q = 7 - 27 + 9 = -11$

   (c) $\lim\limits_{v\to 3} q = 7 + 9 + 1 = 17$

2.

   (a) $\lim\limits_{v\to -1} q = \lim\limits_{v\to -1}(v+2) \cdot \lim\limits_{v\to -1}(v-4) = 1(-4) = -4$

   (b) $\lim\limits_{v\to 0} q = 2(-3) = -6$

   (c) $\lim\limits_{v\to 5} q = 7(2) = 14$

3.

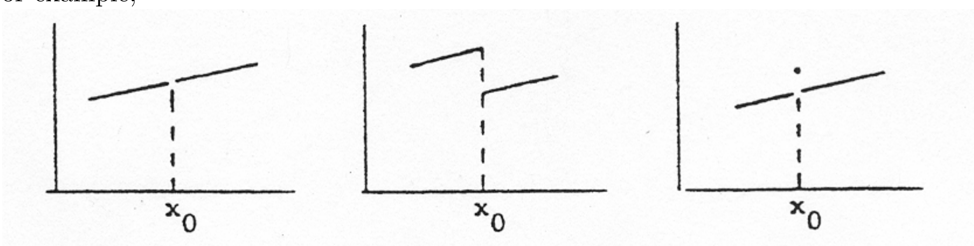
   (a) $\lim\limits_{v\to 0} = \lim\limits_{v\to 0}(3v+5)/\lim\limits_{v\to 0}(v+2) = 5/2 = 2\frac{1}{2}$

(b) $\lim_{v \to 5} q = (15+5)/(5+2) = 20/7 = 2\frac{6}{7}$

(c) $\lim_{v \to -1} q = (-3+5)/(-1+2) = 2/1 = 2$

Exercise 6.7

1. For example,



2. (a) $\lim_{v \to N} q = N^2 - 5N - 2 = g(N)$ (b) Yes. (c) Yes.

3. (a) $\lim_{v \to N} q = (N+2)/(N^2+2) = g(N)$

   (b) Yes. (c) The function is continuous in the domain

4. (a) No. (b) No, because $f(x)$ is not defined at $x = 4$; i.e., $x = 4$ is not in the domain of the function.

   (c) for $x \neq 4$, the function reduces to $y = x - 5$, so $\lim_{x \to 4} y = -1$.

5. No, because $q = v + 1$, as such, is defined at every value of v, whereas the given rational function is *not* defined at $v = 2$ and $v = -2$. The only permissible way to rewrite is to qualify the equation $q = v + 1$ by the restrictions $v \neq 2$ and $v \neq -2$.

6. Yes; each function is not only continuous but also smooth.

CHAPTER 7

Exercise 7.1

1. (a) $dy/dx = 12x^{11}$ (b) $dy/dx = 0$ (c) $dy/dx = 35x^4$
   (d) $dw/du = -3u^{-2}$ (e) $dw/du = -2u^{-1/2}$ (f) $dw/du = u^{-3/4}$

2. (a) $4x^{-5}$ (b) $3x^{-2/3}$ (c) $20w^3$
   (d) $2cx$ (e) abu^{b-1} (f) abu^{-b-1}

3.
   (a) $f'(x) = 18$; thus $f'(1) = 18$ and $f'(2) = 18$.

   (b) $f'(x) = 3cx^2$; thus $f'(1) = 3c$ and $f'(2) = 12c$.

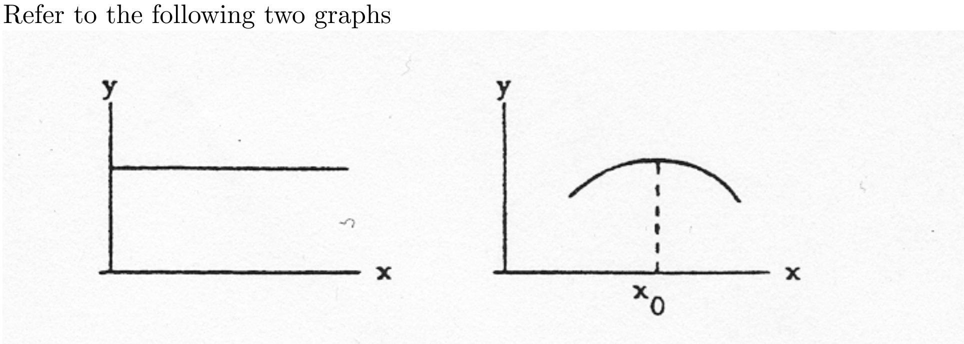
   (c) $f'(x) = 10x^{-3}$; thus $f'(1) = 10$ and $f'(2) = \frac{10}{8} = 4\frac{1}{4}$

   (d) $f'(x) = x^{1/3} = \sqrt[3]{x}$; thus $f'(1) = 1$ and $f'(2) = \sqrt[3]{2}$

   (e) $f'(w) = 2w^{-2/3}$; thus $f'(1) = 2$ and $f'(2) = 2 \cdot 2^{-2/3} = 2^{1/3}$

   (f) $f'(w) = \frac{1}{2}w^{-7/6}$; thus $f'(1) = \frac{1}{2}$ and $f'(2) = \frac{1}{2}(2^{-7/6}) = 2^{-1} \cdot 2^{-7/6}$

4. Refer to the following two graphs



Exercise 7.2

1. $VC = Q^3 - 5Q^2 + 12Q$. The derivative $\frac{d}{dQ}VC = 3Q^2 - 10Q + 12$ is the MC function.

2. $C = AC \cdot Q = Q^3 - 4Q^2 + 174Q$. Thus $MC = dC/dQ = 3Q^2 - 9Q + 174$.
   Since the total-cost function shows zero fixed cost, the situation depicted is the long run.

3. (a) $3(27x^2 + 6x - 2)$ (b) $54x^2 + 78x - 70$

   (c) $12x(x+1)$ (d) $cx(ax - 2b)$

   (e) $-x(9x + 14)$ (f) $2 - \dfrac{x^2+3}{x^2} = \dfrac{x^2-3}{x^2}$

4. (b) $R = AR \cdot Q = 60Q - 3Q^2$, and $MR = dr/dQ = 60 - 6Q$.

   (c) It should.

   (d) The MR curve is twice as steep as the AR curve.

5. Let the average curve be represented by $A = a + bx$. Then the total curve will be $T = A \cdot x = ax + bx^2$, and the marginal curve will be $M = dT/dx = a + bx$.

6. Let $\phi(x) \equiv g(x)h(x)$; this implies that $\phi'(x) = g'(x)h(x) + g(x)h'(x)$. Then we may write

$$\begin{aligned}
\frac{d}{dx}\left[f(x)g(x)h(x)\right] &= \frac{d}{dx}\left[f(x)\phi(x)\right] = f'(x)\phi(x) + f(x)\phi'(x) \\
&= f'(x)g(x)h(x) + f(x)\left[g'(x)h(x) + g(x)h'(x)\right] \\
&= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)
\end{aligned}$$

7. (a) $\dfrac{x^2-3}{x^2}$ (b) $-\dfrac{9}{x^2}$ (c) $\dfrac{30}{(x+5)^2}$ (d) $\dfrac{acx^2 + 2adx - bc}{(cx+d)^2}$

8. (a) $\dfrac{d}{dx}(ax+b) = a$ (b) $\dfrac{d}{dx}x(ax+b) = 2ax + b$

   (c) $\dfrac{d}{dx}\dfrac{1}{ax+b} = \dfrac{-a}{(ax+b)^2}$ (d) $\dfrac{d}{dx}\dfrac{ax+b}{x} = \dfrac{-b}{x^2}$

9.

   (a) Yes; the continuity of $f(x)$ is a necessary condition for $f(x)$ to be differentiable.

   (b) No; a continuous function may not have a continuous derivative function (e.g., Fig. 7.1c).

10.

   (a) $MC = \frac{dTC}{dQ} = 6Q + 7$

   $AC = \frac{TC}{Q} = 3Q + 7 + \frac{12}{Q}$

   (b) $MR = \frac{dTR}{dQ} = 10 - 2Q$

   $AR = \frac{TR}{Q} = 10 - Q$

   (c) $MP = \frac{dTP}{dL} = a + 2bL - cL^2$

   $AP = \frac{TP}{L} = a + bL - cL^2$

Exercise 7.3

1. $dy/dx = (dy/du)(du/dx) = (3u^2 + 2((-2x) = -2x[3(5 - x^2)^2 + 2]$

2. $dw/dx = (dw/dy)(dy/dx) = 2ay(2bx + c) = 2ax(2b^2\mathrm{x}^2 + 3bcx + c^2)$

3.

   (a) Let $w = 3x^2 - 13$; this implies that $dw/dx = 6x$. Since $y = w^3$, we have $\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = 3w^2(6x) = 18x(3x^2 - 13)^2$

   (b) $\frac{dy}{dx} = 189x^2(7x^3 - 5)^8$

   (c) $\frac{dy}{dx} = 5a(ax + b)^4$

4. Both methods yield the same answer $dy/dx = -32(16x + 3)^{-3}$

5. The inverse function is $x = \frac{y}{7} - 3$. The derivatives are $dy/dx = 7$ and $dx/dy = 1/7$; thus the inverse function rule is verified.

6.

   (a) Since $x > 0$, we have $dy/dx = -6x^5 < 0$ for all admissible values of x. Thus the function is strictly decreasing, and dx/dy is equal to $-1/6x^5$, the reciprocal of dy/dx.

   (b) $dy/dx = 20x^4 + 3x^2 + 3 > 0$ for any value of x; thus the function is strictly increasing, and $dx/dy = 1/(20x^4 + 3x^2 + 3)$.

Exercise 7.4

1. (a) $\partial y/\partial x_1 = 6x_1^2 - 22x_1x_2$ $\quad$ $\partial y/\partial x_2 = -11x_1^2 + 6x_2$
   (b) $\partial y/\partial x_1 = 7 + 6x_2^2$ $\quad$ $\partial y/\partial x_2 = 12x_1x_2 - 27x_2^2$
   (c) $\partial y/\partial x_1 = (2(x_2 - 2)$ $\quad$ $\partial y/\partial x_2 = 2x_1 + 3$
   (d) $\partial y/\partial x_1 = 5/(x_2 - 2)$ $\quad$ $\partial y/\partial x_2 = -(5x_1 + 3)/(x_2 - 2)^2$

2. (a) $f_x = 3x^2 + 5y$ $\quad$ $f_y = 5x - 3y^2$
   (b) $f_x = 3x^2 - 4x - 3y$ $\quad$ $f_y = -3(x - 2)$
   (c) $f_x = 5y/(x + y)^2$ $\quad$ $f_y = -5x/(x + y)^2$
   (d) $f_x = (x^2 + 1)/x^2y$ $\quad$ $f_y = -(x^2 - 1)/xy^2$

3. (a)12 (b) -7 (c) 10/9 (d) 1

4. $MPP_K = (0.3)96K^{-0.7}L^{0.7}$ $MPP_L(0.7)96K^{0.3}L^{-0.3}$

5. (a) $U_1 = 2(x_1+2)(x_2+3)^3$ $U_2 = 3(x_1+2)^2(x_2+3)^2$
   (b) $U_1(3,3) = 2160$

6.

   (a) Since $M = D + C$, where $C = cD$, it follows that $M = D + cD = (1+c)D$.
   Since $H = C + R = cD + rD = (c+r)D$, we can write $D = \frac{H}{c+r}$. Thus, by substituting out D, we have $M = \frac{(1+c)H}{c+r}$

   (b) $\frac{\partial M}{\partial r} = \frac{-(1+c)H}{(c+r)^2} < 0$. An increase in in r lowers M

   (c) $\frac{\partial M}{\partial c} = \frac{H(c+r)-(1+c)H}{(c+r)^2} = \frac{H(r-1)}{(c+r)^2} < 0$. An increase in c also lowers M

7. (a) grad $f(x,y,z) = (2x, 3y^2, 4z^3)$
   (b) grad $f(x,y,z) = (yz, xz, xy)$

Exercise 7.5

1. $\frac{\partial Q^*}{\partial a} = \frac{d}{b+d} > 0$ $\frac{\partial Q^*}{\partial b} = \frac{-d(a+c)}{(b+d)^2} < 0$
   $\frac{\partial Q^*}{\partial c} = \frac{-b}{b+d} < 0$ $\frac{\partial Q^*}{\partial d} = \frac{b(a+c)}{(b+d)^2} > 0$

2.

$$\begin{aligned}\frac{\partial Y^*}{\partial I_0}\text{(investment multiplier)} &= \frac{\partial Y^*}{\partial \alpha}\text{(consumption multiplier)} \\ &= \frac{1}{1-\beta+\beta\delta} > 0\end{aligned}$$

$$\begin{aligned}\frac{\partial Y^*}{\partial \beta} &= \frac{-\gamma+(1-\delta)(\alpha+I_0+G_0)}{(1-\beta+\beta\delta)^2} \\ &= \frac{-\gamma+(1-\delta)Y^*}{(1-\beta+\beta\delta)^2} \text{ [by (7.18)]} \\ &= \frac{Y^*-T^*}{(1-\beta+\beta\delta)^2} \text{ [by (7.17)]}\end{aligned}$$

Assuming non-confiscationary taxation, we can take $\frac{\partial Y^*}{\partial \beta}$ to be positive; an increase in the marginal propensity to consume raises the equilibrium income.

3.

(a) Nine.

(b)
$$\frac{\partial x_1^*}{\partial d_1} = \frac{0.66}{0.384} \qquad \frac{\partial x_1^*}{\partial d_2} = \frac{0.30}{0.384} \qquad \frac{\partial x_1^*}{\partial d_3} = \frac{0.24}{0.384}$$
$$\frac{\partial x_2^*}{\partial d_1} = \frac{0.34}{0.384} \qquad \frac{\partial x_2^*}{\partial d_2} = \frac{0.62}{0.384} \qquad \frac{\partial x_2^*}{\partial d_3} = \frac{0.24}{0.384}$$
$$\frac{\partial x_3^*}{\partial d_1} = \frac{0.21}{0.384} \qquad \frac{\partial x_3^*}{\partial d_2} = \frac{0.27}{0.384} \qquad \frac{\partial x_3^*}{\partial d_3} = \frac{0.60}{0.384}$$

or
$$\frac{\partial x^*}{\partial d_1} = \frac{1}{0.384}\begin{bmatrix} 0.66 \\ 0.34 \\ 0.21 \end{bmatrix} \qquad \frac{\partial x^*}{\partial d_2} = \frac{1}{0.384}\begin{bmatrix} 0.30 \\ 0.62 \\ 0.27 \end{bmatrix} \qquad \frac{\partial x^*}{\partial d_3} = \frac{1}{0.384}\begin{bmatrix} 0.24 \\ 0.24 \\ 0.60 \end{bmatrix}$$

Exercise 7.6

1.

(a) $|J| = \begin{vmatrix} 6x_1 & 1 \\ (36x_1^3 + 12x_1x_2 + 48x_1) & (6x_1^2 + 2x_2 + 8) \end{vmatrix} = 0$

The function is dependent.

(b) $|J| = \begin{vmatrix} 6x_1 & 4x_2 \\ 5 & 0 \end{vmatrix} = -20x_2$

Since $|J|$ is not identically zero, the functions are independent

2.

(a) $|J| = \begin{vmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{vmatrix} = |V|$

(b) Since V has an inverse matrix $(I - A)$, it must be nonsingular, and so $|V| \neq 0$, or $|J| \neq 0$. The equations in (7.22) are thus functionally independent.

CHAPTER 8

Exercise 8.1

1.

(a) $dy = -3\left(x^2+1\right)\,dx$

(b) $dy = (14x - 51)\,dx$

(c) $dy = \dfrac{1-x^2}{\left(x^2+1\right)^2}\,dx$

2. $\epsilon_{MY} = \dfrac{\frac{dM}{dY}}{\frac{M}{Y}} = \dfrac{\text{marginal propensity to import}}{\text{average propensity to import}}$

3.

(a) $\dfrac{dC}{dY} = b \qquad \dfrac{C}{Y} = \dfrac{a+bY}{Y}$

(b) $\epsilon_{CY} = \dfrac{\frac{dC}{dY}}{\frac{C}{Y}} = \dfrac{bY}{a+bY} > 0$

(c) Since $bY < a + bY$, it follows that $\epsilon_{CY} < 1$.

4. Since $Q = kP^{-n}$, with $\frac{dQ}{dP} = -nkP^{n-1}$ and $\frac{Q}{P} = kP^{n-1}$, the point elasticity of demand is $\epsilon_d = -n =$ a constant.

(a) No.

(b) When $n = 1$, the demand function is $Q = \frac{k}{P}$, which plots as a rectangular hyperbola, with a unitary point elasticity everywhere.

5.

(a) Any positively sloped straight line emanating from the point of origin will do. [see the broken line in Fig. 8.3b.]

(b) The equation for such a line is $y = bx$ (with zero vertical intercept), so that $dy/dx = b = y/x$. Hence, by (8.6), the elasticity is 1, a constant.

6. (a) the price elasticity of demand is

$$\in^d = \frac{\partial Q}{\partial P}\left(\frac{P}{Q}\right)$$

where the partial derivative with respect to price is

$$\frac{\partial Q}{\partial P} = -2$$

and $Q = 100 - 2(20) + 0.02(5000) = 160$.Therefore

$$\in^d = (-2)\,\frac{20}{160} = -\frac{1}{4}$$

(b) The income elasticity of demand is

$$\eta = \frac{\partial Q}{\partial Y}\left(\frac{Y}{Q}\right)$$

where the partial derivative with respect to income is

$$\frac{\partial Q}{\partial Y} = 0.02$$

Substituting the relevant values

$$\eta = (0.02)\frac{5000}{160} = 0.625$$

Exercise 8.2

1. Let ∇U be the row vector $[U_1, \ldots, U_n]$, and dx be the column vector $[dx_1, \ldots, dx_n]$. Then $dU = \nabla U dx$.

2.

   (a) $dz = (6x + y)\ dx + \left(x - 6y^2\right)\ dy$

   (b) $dU = (2 + 9x_2)\ dx_1 + (9x_1 + 2x_2)\ dx_2$

3.

   (a) $dy = \dfrac{x_2}{(x_1 + x_2)^2}\,dx_1 - \dfrac{x_1}{(x_1 + x_2)^2}dx_2$

   (b) $dy = 2\left(\frac{x_2}{x_1+x_2}\right)^2 dx_1 + 2\left(\frac{x_1}{x_1+x_2}\right)^2\ dx_2$

4.

$$\begin{aligned}\frac{\partial Q}{\partial P} &= 2bP, \text{ thus } \epsilon_{QP} = 2bP\frac{P}{Q} = \frac{2bP^2}{a+bP^2+R^{1/2}};\\ \frac{dQ}{dR} &= \frac{1}{2}R^{-1/2}, \text{ thus } \epsilon_{QR} = \frac{1}{2}R^{-1/2}\frac{R}{Q} = \frac{R^{1/2}}{2\left(a+bP^2+R^{1/2}\right)}.\end{aligned}$$

5.

$$\begin{aligned}\frac{\partial}{\partial P}\epsilon_{QP} &= \frac{4bP\left(a+R^{1/2}\right)}{\left(a+bP^2+R^{1/2}\right)^2} \gtreqless 0 \text{ as } \left(a+R^{1/2}\right) \gtreqless 0\\ \frac{\partial}{\partial R}\epsilon_{QP} &= \frac{-bP^2R^{-1/2}}{\left(a+bP^2+R^{1/2}\right)^2} < 0\\ \frac{\partial}{\partial P}\epsilon_{QR} &= \frac{-bPR^{-1/2}}{\left(a+bP^2+R^{1/2}\right)^2} < 0\\ \frac{\partial}{\partial R}\epsilon_{QR} &= \frac{R^{1/2}\left(a+bP^2\right)}{4\left(a+bP^2+R^{1/2}\right)^2} \gtreqless 0 \text{ as } \left(a+bP^2\right) \gtreqless 0\end{aligned}$$

Each of these derivatives adheres to a single sign, thus each elasticity varies with P and R in a strictly monotonic function. (Note that even $\frac{\partial}{\partial P}\epsilon_{QP}$ adheres to a single sign, because in the context of that derivative, R is a constant, so that $\left(a+R^{1/2}\right)$ has a single magnitude with a single sign. The same reasoning applies also to $\frac{\partial}{\partial R}\epsilon_{QR}$.)

6. $\epsilon_{XP} = \dfrac{\frac{\partial X}{\partial P}}{\frac{X}{P}} = \dfrac{-2P^{-3}}{Y_f^{1/2}P^{-1}+P^{-3}} = \dfrac{-2}{Y_f^{1/2}P^2+1}$

7.

(a)

$$\begin{aligned}U_x &= 15x^2 - 12y\\ U_y &= -12x - 30y^4\\ dU &= (15x^2-12y)dx - (12x-30y^4)dy\end{aligned}$$

(b)

$$\begin{aligned}U_x &= 14xy^3\\ U_y &= 21x^2y^2\\ dU &= (14xy^3)dx + (21x^2y^2)dy\end{aligned}$$

(c)

$$\begin{aligned} U_x &= 3x^2(8) + (8x - 7y)(6x) \\ U_y &= 3x^2(-7) + (8x - 7y)(0) \\ dU &= (72x^2 - 42xy)dx - 21x^2 dy \end{aligned}$$

(d)

$$\begin{aligned} U_x &= (5x^2 + 7y)(2) + (2x - 4y^3)(10x) \\ U_y &= (5x^2 + 7y)(-12y^2) + (2x - 4y^3)(7) \\ dU &= (30x^2 - 40xy^3 + 14y)dx - (112y^3 + 60x^2y^2 - 14x)dy \end{aligned}$$

(e)

$$\begin{aligned} U_x &= \frac{(x-y)(0) - 9y^3(1)}{(x-y)^2} \\ U_y &= \frac{(x-y)(27y^2) - 9y^3(-1)}{(x-y)^2} \\ dU &= \frac{-9y^3}{(x-y)^2}dx + \frac{27xy^2 - 18y^3}{(x-y)^2}dy \end{aligned}$$

(f)

$$\begin{aligned} U_x &= 3(x - 3y)^2(1) \\ U_y &= 3(x - 3y)^2(-3) \\ dU &= 3(x - 3y)^2 dx - 9(x - 3y)^2 dy \end{aligned}$$

Exercise 8.3

1.

(a) $dz = 6x\,dx + (y\,dx + x\,dy) - 6y^2\,dy = (6x + y)\;dx + \left(x - 6y^2\right)\;dy$

(b) $dU = 2\,dx_1 + (9x_2\,dx_1 + 9x_1\,dx_2) + 2x_2\,dx_2 = (2 + 9x_2)\;dx_1 + (9x_1 + 2x_2)\;dx_2$

2.

(a) $dy = \frac{(x_1+x_2)\,dx_1 - x_1(dx_1+dx_2)}{(x_1+x_2)^2} = \frac{x_2 dx_1 - x_1 dx_2}{(x_1+x_2)^2}$

(b) $dy = \frac{(x_1+x_2)(2x_2\,dx_1+2x_1\,dx_2)-2x_1x_2(dx_1+dx_2)}{(x_1+x_2)^2} = \frac{2x_2^2dx_1+2x_1^2dx_2}{(x_1+x_2)^2}$

3.

(a) $dy = 3\left[(2x_2-1)(x_3+5)\ dx_1 + 2x_1(x_3+5)\ dx_2 + x_1(2x_2-1)\ dx_3\right]$

(b) $dy = 3(2x_2-1)(x_3+5)\ dx_1$

4. Rule II: $d(cu^n) = \left(\frac{d}{du}cu^n\right)\ du = cnu^{n-1}\,du$

Rule III: $d(u \pm v) = \frac{\partial(u\pm v)}{\partial u}\,du + \frac{\partial(u\pm v)}{\partial v}dv = 1\,du + (\pm 1)\,dv = du \pm dv$

Rule IV: $d(uv) = \frac{\partial(uv)}{\partial u}\,du + \frac{\partial(uv)}{\partial v}\,dv = v\,du + u\,dv$

Rule V: $d\left(\frac{u}{v}\right) = \frac{\partial(u/v)}{\partial u}\,du + \frac{\partial(u/v)}{\partial v}\,dv = \frac{1}{v}\,du - \frac{u}{v^2}dv = \frac{1}{v^2}(v\,du - u\,dv)$

Exercise 8.4

1.

(a) $\frac{dz}{dy} = z_x\frac{dx}{dy} + z_y = (5+y)6y + x - 2y = 28y + 6y^2 + x = 28y + 9y^2$

(b) $\frac{dz}{dy} = 4y - \frac{8}{y^3}$

(c) $\frac{dz}{dy} = -15x + 3y = 108y - 30$

2.

(a) $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2x-8y)(3) + \left(-8x-3y^2\right)(-1) = 14x - 24y + 3y^2 = 3t^2 + 60t - 21$

(b) $\frac{dz}{dt} = 7(4t) + t(1) + v = 29t + v = 30t + 1$

(c) $\frac{dz}{dt} = bf_x + kf_y + f_t$

3. $\frac{dQ}{dt} = a\alpha AK^{\alpha-1}L^{\beta} + b\beta AK^{\alpha}L^{\beta-1} + A'(t)K^{\alpha}L^{\beta} = \left[a\alpha\frac{A}{K} + b\beta\frac{A}{L} + A'(t)\right]K^{\alpha}L^{\beta}$

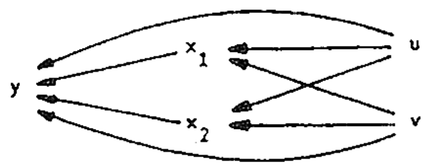
4.

(a)

$$\begin{aligned}
\frac{\oint W}{\oint u} &= \frac{\partial W}{\partial x}\frac{dx}{du} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial W}{\partial u} = (2ax+by)(\alpha) + (bx)(\gamma) + c \\
&= \alpha\left[2a(\alpha u + \beta v) + b\gamma u\right] + b\gamma(\alpha u + \beta v) + c \\
\frac{\oint W}{\oint v} &= \frac{\partial W}{\partial x}\frac{dx}{dv} + \frac{\partial W}{\partial y}\frac{\partial y}{\partial v} = (2ax+by)(\beta) + (bx)(0) \\
&= \beta\left[2a(\alpha u + \beta v) + b\gamma u\right]
\end{aligned}$$

(b) $\frac{\oint W}{\oint u} = 10uf_1 + f_2 \qquad \frac{\oint W}{\oint v} = 3f_1 - 12v^2 f_2$

5.



6. $\frac{\oint y}{\oint v} = \frac{\partial y}{\partial x_1}\frac{\partial x_1}{\partial v} + \frac{\partial y}{\partial x_2}\frac{\partial x_2}{\partial v} + \frac{\partial y}{\partial v}$

Exercise 8.5

1.

(a) $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(-6)}{1} = 6$

(b) $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(-12)}{3} = 4$

(c) $\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-(2x+6)}{-1} = 2x + 6$

2.

(a) $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{6x+2y}{12y^2+2x}$

(b) $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{60x^4}{-2} = 30x^4$

(c) $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{14x+2y^2}{36y^3+4xy}$

(d) $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{18x^2}{-3} = 6x^2$

3.

(a)

$$\begin{aligned}\frac{dy}{dx} &= -\frac{fx}{fy} = -\frac{2xy^3 + yz}{3x^2y^2 + xz} \\ \frac{dy}{dz} &= -\frac{fz}{fy} = -\frac{2z + xy}{3x^2y^2 + xz}\end{aligned}$$

(b)

$$\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{3x^2z^2+4yz}{3y^2+4xz}$$
$$\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2x^3z+4xy}{3y^2+4xz}$$

(c)

$$\frac{dy}{dx} = -\frac{fx}{fy} = -\frac{6xy^3+z^2y^2+4y^3zx^3}{9x^2y^2+2xz^2y+3y^2zx^4+2yz}$$
$$\frac{dy}{dz} = -\frac{fz}{fy} = -\frac{2xzy^2+y^3x^4+y^2}{9x^2y^2+2xz^2y+3y^2zx^4+2yz}$$

4.

(a) $\dfrac{\partial U}{\partial x_2} = \dfrac{-\frac{\partial F}{\partial x_2}}{\frac{\partial F}{\partial U}}, \qquad \dfrac{\partial U}{\partial x_n} = -\dfrac{\frac{\partial F}{\partial x_n}}{\frac{\partial F}{\partial U}}, \qquad \dfrac{\partial x_3}{\partial x_2} = -\dfrac{\frac{\partial F}{\partial x_2}}{\frac{\partial F}{\partial x_3}}, \qquad \dfrac{\partial x_4}{\partial x_n} = -\dfrac{\frac{\partial F}{\partial x_n}}{\frac{\partial F}{\partial x4}}$

(b) The first two are marginal utilities; the last two are slopes of indifference curves (negatives of marginal rates of substitution).

5.

(a) Point (y=3, x=1) does satisfy the given equation. Moreover, $F_x = 3x^2 - 4xy + 3y^2$ and $F_y = -2x^2 + 6xy$ are continuous, and $F_y = 16 \neq 0$ at the given point. Thus an implicit function is defined, with:

$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2-4xy+3y^2}{-2x^2+6xy} = -\frac{18}{16} = -\frac{9}{8}$ at the given point

(b) The given point satisfies this equation also. Since both $F_x = 4x + 4y$ and $F_y = 4x - 4y^3$ are continuous, and $F_y = -104 \neq 0$ at the given point, an implicit function is again defined.

$\frac{dy}{dx} = -\frac{4x+4y}{4x-4y^3} = -\frac{16}{-104} = -\frac{2}{13}$ at the given point

6. Point $(x = 1, y = 2, z = 0)$ satisfies the given equation. Since the three derivatives $F_x = 2x + 3y$, $F_y = 3x + 2z + 2y$, $F_z = 2y + 2z$ are all continuous, and $F_z = 4 \neq 0$ at the given point, an implicit function $z = f(x, y)$ is defined. At the given point, we have

$\frac{\partial z}{\partial x} = -\frac{2x+3y}{2y+2z} = -2 \qquad \frac{\partial z}{\partial y} = -\frac{3x+2z+2y}{2y+2z} = -\frac{7}{4}$

7. The given equation can be solved for y, to yield the function $y = x$ (with the 45^0 line as its graph). Yet, at the point $(0, 0)$, which satisfies the given equation and is on the 45^0 line,

we find $F_y = -3(x-y)^2 = 0$, which violates the condition of a nonzero F_y as cited in the theorem. This serves us to show that this condition is not a necessary condition for the function $y = f(x)$ to be defined.

8. By (8.23), $\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = \left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right) = -1$.

9. At least one of the partial derivatives in the vector of constants in (8.28') must be nonzero; otherwise, the variable x_1 does not affect F^1, F^2 and F^3, and has no legitimate status as an argument in the F functions in (8.24).

10. To find the nonincome-tax multiplier $\frac{\partial Y^*}{\partial \gamma}$ (along with $\frac{\partial C^*}{\partial \gamma}$ and $\frac{\partial T^*}{\partial \gamma}$), the relevant matrix equation is

$$\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial Y^*}{\partial \gamma} \\ \frac{\partial C^*}{\partial \gamma} \\ \frac{\partial T^*}{\partial \gamma} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F^1}{\partial \gamma} \\ -\frac{\partial F^2}{\partial \gamma} \\ -\frac{\partial F^3}{\partial \gamma} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

$$\frac{\partial Y^*}{\partial \gamma} = \frac{\begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & \beta \\ 1 & 0 & 1 \end{vmatrix}}{|J|} = \frac{-\beta}{1-\beta+\beta\delta} \qquad \text{[by (8.31)]}$$

This result does check with (7.20).

Exercise 8.6

1.

(a) S'=marginal propensity to save; T'=marginal income tax rate; I'=marginal propensity to invest.

(b) Writing the equilibrium condition as $F(Y; G_0) = S(Y) + T(Y) - I(Y) - G_0 = 0$, we find that F has continuous partial derivatives and $\frac{\partial F}{\partial Y} = S' + T' - I' \neq 0$. Thus the implicit-function theorem is applicable. The equilibrium identity is: $S(Y^*) + T(Y^*) - I(Y^*) - G_0 \equiv 0$.

(c) By the implicit-function rule, we have

$$\left(\frac{dY^*}{dG_0}\right) = -\frac{-1}{S'+T'-I'} = \frac{1}{S'+T'-I'} > 0$$

As increase in G_0 will increase the equilibrium national income.

2.

(a) $F(P; Y_0, T_0) = D(P, Y_0) - S(P, T_0) = 0$

(b) F has continuous partial derivatives, and $F_P = D_P - S_P \neq 0$, thus the implicit-function theorem is applicable. The equilibrium identity is: $D(P^*, Y_0) - S(P^*, T_0) \equiv 0$.

(c) By the implicit-function rule,

$\left(\frac{\partial P^*}{\partial Y_0}\right) = -\frac{D_{y_0}}{D_{P^*} - S_{P^*}} > 0 \qquad \left(\frac{\partial P^*}{\partial T_0}\right) = -\frac{-S_{T_0}}{D_{P^*} - S_{P^*}} > 0$

An increase in income or taxes will raise the equilibrium price.

(d) The supply function implies $Q^* = S(P^*, T_0)$; thus $\left(\frac{\partial Q^*}{\partial Y_0}\right) = \frac{\partial S}{\partial P^*}\left(\frac{\partial P^*}{\partial Y_0}\right) > 0$.

The demand function implies $Q^* = D(P^*, Y_0)$; thus $\left(\frac{\partial Q^*}{\partial T_0}\right) = \frac{\partial D}{\partial P^*}\left(\frac{\partial P^*}{\partial T_0}\right) < 0$.

<u>Note</u>: To use the demand function to get $\left(\frac{\partial Q^*}{\partial Y_0}\right)$ would be more complicated, since Y_0 has both direct and indirect effects on Q_d^*. A similar complication arises when the supply function is used to get the other comparative-static derivative.

3. Writing the equilibrium conditions as

$$\begin{aligned} F^1(P, Q; Y_0, T_0) &= D(P, Y_0) - Q = 0 \\ F^2(P, Q; Y_0, T_0) &= S(P, T_0) - Q = 0 \end{aligned}$$

We find $|J| = \begin{vmatrix} D_P & -1 \\ S_P & -1 \end{vmatrix} = S_P - D_P \neq 0$. Thus the implicit-function theorem still applies, and we can write the equilibrium identities

$$\begin{aligned} D(P^*, Y_0) - Q^* &\equiv 0 \\ S(P^*, T_0) - Q^* &\equiv 0 \end{aligned}$$

Total differentiation yields

$$\begin{aligned} D_{P^*} dP^* - dQ^* &= -D_{Y_0} dY_0 \\ S_{P^*} dP^* - dQ^* &= -S_{T_0} dT_0 \end{aligned}$$

When Y_0 is disequilibrating factor ($dT_0 = 0$), we have

$$\begin{bmatrix} D_{P^*} & -1 \\ S_{P^*} & -1 \end{bmatrix} \begin{bmatrix} \left(\frac{\partial P^*}{\partial Y_0}\right) \\ \left(\frac{\partial Q^*}{\partial Y_0}\right) \end{bmatrix} = \begin{bmatrix} -D_{Y_0} \\ 0 \end{bmatrix}$$

Thus $\left(\frac{\partial P^*}{\partial Y_0}\right) = \frac{D_{Y_0}}{S_{P^*}-D_{P^*}} > 0$ and $\left(\frac{\partial Q^*}{\partial Y_0}\right) = \frac{D_{Y_0}S_{P^*}}{S_{P^*}-D_{P^*}} > 0$

When T_0 is the disequilibrium factor ($dY_0 = 0$), we can similarly get $\left(\frac{\partial P^*}{\partial T_0}\right) = \frac{-S_{T_0}}{S_{P^*}-D_{P^*}} > 0$ and $\left(\frac{\partial Q^*}{\partial T_0}\right) = \frac{-S_{T_0}D_{P^*}}{S_{P^*}-D_{P^*}} < 0$

4.

(a) $\partial D/\partial P < 0$, and $\partial D/\partial t_0 > 0$

(b) $F(P; t_0, Q_{s0}) = D(P, t_0) - Q_{s0} = 0$

(c) Since the partial derivatives of F are all continuous, and $F_P = \frac{\partial D}{\partial P} \neq 0$, the implicit-function theorem applies.

(d) To find $\left(\frac{\partial P^*}{\partial t_0}\right)$, use the implicit-function rule on the equilibrium identity $D(P^*, t_0) - Q_{s0} \equiv 0$, to get

$$\left(\frac{\partial P^*}{\partial t_0}\right) = -\frac{\frac{\partial D}{\partial t_0}}{\frac{\partial D}{\partial P^*}} > 0$$

An increase in consumers' taste will raise the equilibrium price.

5.

(a) Yes.

(b) $kY + L(i)$

(c) We can take the two equilibrium conditions as the equilibrium $F^1 = 0$ and $F^2 = 0$, respectively. Since the Jacobian is nonzero:

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial Y} & \frac{\partial F^1}{\partial i} \\ \frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial i} \end{vmatrix} = \begin{vmatrix} 1-C' & -I' \\ k & L' \end{vmatrix} = L'(1-C') + kI' < 0$$

the implicit-function theorem applies, and we have the equilibrium identities

$$\begin{aligned} Y^* - C(Y^*) - I(i^*) - G_0 &\equiv 0 \\ kY^* + L(i^*) - M_{s0} &\equiv 0 \end{aligned}$$

with M_{s0} as the disequilibrating factor, we can get the equation

$$\begin{bmatrix} 1-C' & -I' \\ k & L' \end{bmatrix} \begin{bmatrix} \frac{\partial Y^*}{\partial G_0} \\ \frac{\partial i^*}{\partial G_0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This yields the results

$$\left(\frac{\partial Y^*}{\partial G_0}\right) = \frac{L'}{|J|} > 0 \quad \text{and} \quad \left(\frac{\partial i^*}{\partial G_0}\right) = -\frac{k}{|J|} > 0$$

(d) $f'(x) = 6x - 6 = 0$ iff $x = 1$; $f(1) = -1$ is a relative minimum.

6.

(a) The first equation stands, but the second equation should be changed to

$$kY - M_{s0} = 0 \quad or \quad kY + L_0 - M_{s0}$$

(b)

$$|J|' = \begin{vmatrix} 1 - C' & -I' \\ k & 0 \end{vmatrix} = kI'$$

$|J|'$ has a smaller numerical value than $|J|$.

(c) Yes.

(d) With M_{s0} changing, we have

$$\begin{bmatrix} 1 - C' & -I' \\ k & 0 \end{bmatrix} \begin{bmatrix} (\partial Y^*/\partial M_{s0}) \\ (\partial i^*/\partial M_{s0}) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus $(\partial Y^*/\partial M_{s0}) = I'/|J|' > 0$ and $(\partial i^*/\partial M_{s0}) = (1 - C')/|J|' < 0$.

Next, with G_0 changing, we have

$$\begin{bmatrix} 1 - C' & -I' \\ k & 0 \end{bmatrix} \begin{bmatrix} (\partial Y^*/\partial G_0) \\ (\partial i^*/\partial G_0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus $(\partial Y^*/\partial G_0) = 0$ and $(\partial i^*/\partial M_{s0}) = -k/|J|' > 0$

(e) Fiscal policy becomes totally ineffective in the new model.

(f) Since $|J|'$ is numerically smaller than $|J|$, we find that $I'/|J|' > I'/|J|$. Thus monetary policy becomes more effective in the new model.

CHAPTER 9

Exercise 9.2

1. (a) $f'(x) = -4x + 8 = 0$ iff $x = 2$; the stationary value $f(2) = 15$ is a relative maximum.
   (b) $f'(x) = 10x + 1 = 0$ iff $x = -1/10$; $f(-1/10) = -1/20$ is a relative minimum.
   (c) $f'(x) = 6x = 0$ iff $x = 0$; $f(0) = 3$ is a relative minimum.

2.
   (a) Setting $f'(x) = 3x^2 - 3 = 0$ yields two critical values, 1 and -1. The latter is outside the domain; the former leads to $f(1) = 3$, a relative minimum.
   (b) The only critical value is $x^* = 1$; $f(1) = 10\frac{1}{3}$ is a point of inflection.
   (c) Setting $f'(x) = -3x^2 + 9x - 6 = 0$ yields two critical values, 1 and 2; $f(1) = 3.5$ is a relative minimum but $f(2) = 4$ is a relative maximum.

3. When $x = 1$, we have $y = 2$ (a minimum); when $x = -1$, we have $y = -2$ (a maximum). These are in the nature of <u>relative</u> extrema, thus a minimum can exceed a maximum.

4. (a) $M = \phi'(x)$, $A = \phi(x)/x$
   (b) When A reaches a relative extremum, we must have
   $$\frac{dA}{dx} = \frac{1}{x^2}[x\phi'(x) - \phi(x)] = 0$$
   This occurs only when $x\phi'(x) = \phi(x)$, that is, only when $\phi'(x) = \phi(x)/x$, or only when $M = A$.
   (c) The marginal and average curves must intersect when the latter reaches a peak or a trough.
   (d) $\epsilon = \frac{M}{A} = 1$ when $M = A$.

Exercise 9.3

1. (a) $f'(x) = 2ax + b$; $f''(x) = 2a$; $f'''(x) = 0$
   (b) $f'(x) = 28x^3 - 3$; $f''(x) = 84x^2$; $f'''(x) = 168x$

(c) $f'(x) = 3(1-x)^{-2}$; $f''(x) = 6(1-x)^{-3}$; $f'''(x) = 18(1-x)^{-4}$

(d) $f'(x) = 2(1-x)^{-2}$; $f''(x) = 4(1-x)^{-3}$; $f'''(x) = 12(1-x)^{-4}$

2. (a) and (b)

3. (a) An example is a modified version of the curve in Fig. 9.5a, with the arc AB replaced by a line segment AB.

   (b) A straight line.

4. Since $dy/dx = b/(c+x)^2 > 0$, and $d^2y/dx^2 = -2b/(c+x)^3 < 0$, the curve must show y increasing at a decreasing rate. The vertical intercept (where $x = 0$) is $a - \frac{b}{c}$. when x approaches infinity, y tends to the value a, which gives a horizontal asymptote. Thus the range of the function is the interval $[a - \frac{b}{c}, a)$. To use it as a consumption function, we should stipulate that:

   $a > \frac{b}{c}$ [so that consumption is positive at zero income]

   $b > c^2$ [so that $MPC = dy/dx$ is a positive fraction throughout]

5. the function $f(x)$ plots as a straight line, and $g(x)$ plots as a curve with either a peak or a bottom or an inflection point at $x = 3$. In terms of stationary points, every point on $f(x)$ is a stationary point, but the only stationary point on $g(x)$ we know of is at $x = 3$.

   (a) The utility function should have $f(0) - 0$, $f'(x) > 0$, and $f''(x) = 0$ for all x. It plots as an upward-sloping straight line emanating from the point of origin.

   (b) In the present case, the MN line segment would coincide with the utility curve. Thus points A and B lie on top of each other, and $U(15) = EU$.

Exercise 9.4

1. (a) $f'(x) = -4x+8$; $f''(x) = -4$. The critical value is $x^* = 2$; the stationary value $f(2) = 33$ is a maximum.

   (b) $f'(x) = 3x^2 + 12x$; $f''(x) = 6x + 12$. The critical values are 0 and -4. $f(0) = 9$ is a minimum, because $f''(0) = 12 > 0$, but $f''(-4) = 41$ is a maximum, because $f''(-4) = -12 < 0$.

(c) $f'(x) = x^2 - 6x + 5$; $f''(x) = 2x - 6$. The critical values are 1 and 5. $f(1) = 5\frac{1}{3}$ is a maximum because $f''(1) = -4$, but $f(5) = -5\frac{1}{3}$ is a minimum because $f''(5) = 4$.

(d) $f'(x) = 2/(1-2x)^2 \neq 0$ for any value of x; there exists no relative extremum.

2. Excluding the wall side, the other three sides must satisfy $L + 2W = 64$ft, or $L = 64 - 2W$. The area is therefore

$$A = WL = W(64 - 2W) = 64W - 2W^2$$

To maximize A, it is necessary that $dA/dW = 64 - 4W = 0$, which can occur only when $W = 16$. Thus

$$W^* = 16\text{ft} \qquad L^* = 64 - 2W^* = 32\text{ft} \qquad A^* = WL = 512\text{ft}^2$$

Inasmuch as $d^2A/dW^2 = -4$ is negative, A^* is a maximum.

3. (a) Yes.

(b) From the demand function, we first get the AR function $P = 100 - Q$. Then we have $R = PQ = (100 - Q)Q = 100Q - Q^2$.

(c) $\pi = R - C = -\frac{1}{3}Q^3 + 6Q^2 - 11Q - 50$

(d) Setting $d\pi/dQ = -Q^2 + 12Q - 11 = 0$ yields two critical values 1 and 11. Only $Q^* = 11$ gives a maximum profit.

(e) Maximum profit $= 111\frac{1}{3}$

4. If b=0, then the MC-minimizing output level becomes $Q^* = -\frac{b}{3a} = 0$. With its minimum at zero output. The MC curve must be upward-sloping throughout. Since the increasing segment of MC is associated with the convex segment of the C curve, $b = 0$ implies that the C curve will be convex throughout.

5. (a) The first assumption means $\pi(0) < 0$. Since $\pi(0) = k$, we need the restriction $k < 0$.

(b) Strict concavity means $\pi''(Q) < 0$. Since $\pi''(Q) = 2h$, we should have $h < 0$.

(c) The third assumption means $\pi'(Q^*) = 0$, or $2hQ^* + j = 0$. Since $Q^* = -j/2h$, and since $h < 0$, the positivity of Q^* requires that $j > 0$.

6. (a) $Q = f(L)$; $R = P_0Q = P_0f(L)$; $C = W_0L + F$; $\pi = R - C = P_0f(L) - W_0L - F$

(b) $d\pi/dL = P_0 f'(L) - W_0 = 0$, or $P_0 f'(L) = W_0$. The value of marginal product must be equated to the wage rate.

(c) $d^2\pi/dL^2 = P_0 f''(L)$. If $f''(L) < 0$ (diminishing MPP_L), then we can be sure that profit is maximized by L^*.

7. (a) $S = \frac{d}{dQ} AR = -23 + 2.2Q - 0.054Q^2$

(b) $\frac{dS}{dQ} = 2.2 - 0.108Q = 0$ at $Q^* = 20.37$ (approximately); since $\frac{d^2S}{dQ^2} = -0.108 < 0$, Q^* will maximize S.

$$S_{\max} = S|_{Q=Q^*} = -23 + 2.2(20.37) - 0.054(20.37)^2 = -0.59(\text{approximately}).$$

(c) Since $S_{\max}$ is negative, all S values must be negative.

Exercise 9.5

1. (a) 120 (b) 40320 (c) $\frac{4(3!)}{3!} = 4$ (d) $\frac{(6)(5)(4!)}{4!} = 6 \cdot 5 = 30$

(e) $\frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$

2. (a)

$$\begin{aligned}
\phi(x) &= (1-x)^{-1} & \text{so that} \quad \phi(0) &= 1 \\
\phi'(x) &= (1-x)^{-2} & \phi'(0) &= 1 \\
\phi''(x) &= 2(1-x)^{-3} & \phi''(0) &= 2 \\
\phi'''(x) &= 6(1-x)^{-4} & \phi'''(0) &= 6 \\
\phi^{(4)}(x) &= 24(1-x)^{-5} & \phi^{(4)}(0) &= 24
\end{aligned}$$

Thus, according to (9.14), the first five terms are $1 + x + x^2 + x^3 + x^4$

(b)

$$\begin{aligned}
\phi(x) &= (1-x)/(1+x) & \text{so that} \quad \phi(0) &= 1 \\
\phi'(x) &= -2(1+x)^{-2} & \phi'(0) &= -2 \\
\phi''(x) &= 4(1+x)^{-3} & \phi''(0) &= 4 \\
\phi'''(x) &= -12(1+x)^{-4} & \phi'''(0) &= -12 \\
\phi^{(4)}(x) &= 48(1+x)^{-5} & \phi^{(4)}(0) &= 48
\end{aligned}$$

Thus, by (9.14), the first five terms are $1 - 2x + 2x^2 - 2x^3 + 2x^4$

3. (a) $\phi(-2) = 1/3$, $\phi'(-2) = 1/9$, $\phi''(-2) = 2/27$, $\phi'''(-2) = 6/81$, and $\phi^{(4)}(-2) = 24/243$. Thus, by (9.14),

$$\begin{aligned}\phi(x) &= \frac{1}{3} + \frac{1}{9}(x+2) + \frac{1}{27}(x+2)^2 + \frac{1}{81}(x+2)^3 + \frac{1}{243}(x+2)^4 + R_4 \\ &= \frac{1}{243}(211 + 131x + 51x^2 + 11x^3 + x^4) + R_4\end{aligned}$$

(b) $\phi(-2) = -3$, $\phi'(-2) = -2$, $\phi''(-2) = -4$, $\phi'''(-2) = -12$, and $\phi^{(4)}(-2) = -48$. Thus, by (9.14),

$$\begin{aligned}\phi(x) &= -3 - 2(x+2) - 2(x+2)^2 - 2(x+2)^3 - 2(x+2)^4 + R_4 \\ &= -63 - 98x - 62x^2 - 18x^3 - 2x^4 + R_4\end{aligned}$$

4. When $x = x_0$, all the terms on the right of (9.14) except the first one will drop out (including R_n), leaving the result $\phi(x) = \phi(x_0)$.

Exercise 9.6

1. (a) $f'(x) = 3x^2 = 0$ only when $x = 0$, thus $f(0) = 0$ is the only stationary value. The first nonzero derivative value is $f'''(0) = 6$; so $f(0)$ is an inflection point.

   (b) $f'(x) = -4x^3 = 0$ only when $x = 0$. The stationary value $f(0) = 0$ is a relative maximum because the first nonzero derivative value is $f^{(4)}(0) = -24$.

   (c) $f'(x) = 6x^5 = 0$ only when $x = 0$. The stationary value $f(0) = 5$ is a relative minimum since the first nonzero derivative value is $f^{(6)}(0) = 720$.

2. (a) $f'(x) = 3(x-1)^2 = 0$ only when $x = 1$. The first nonzero derivative value is $f'''(1) = 6$. Thus the stationary value $f(1) = 16$ is associated with an inflection point.

   (b) $f'(x) = 4(x-2)^3 = 0$ only when $x = 2$. Since the first nonzero derivative value is $f^{(4)}(2) = 24$, the stationary value $f(2) = 0$ is a relative minimum.

   (c) $f'(x) = -6(3-x)^5 = 0$ only when $x = 3$. Since the first nonzero derivative value is $f^{(6)}(3) = 720$, the stationary value $f(3) = 7$ is a relative minimum.

   (d) $f'x) = -8(5-2x)^3 = 0$ only when $x = 2.5$,. Since the first nonzero derivative value is $f^{(4)}(2.5) = 384$, the stationary value $f(2.5) = 8$ is a relative minimum.

CHAPTER 10

Exercise 10.1

1. (a) Yes.

   (b) Yes, because at $t = 0$, the value of y for the two functions are identical: $3^0 = 1$, and $3^{2(0)} = 1$.

   (a) Yes.

   (b) No, because at $t = 0$, the value of y for the two functions are unequal: $4^0 = 1$, but $3\left(4^0\right) = 3$.

2. (a) Let $w = 5t$ (so that $dw/dt = 5$), then $y = e^w$ and $dy/dw = e^w$. Thus,by the chain rule, $\frac{dy}{dt} = \frac{dy}{dw}\frac{dw}{dt} = 5e^w = 5e^{5t}$.

   (b) Let $w = 3t$, then $y = 4e^w$ and $dy/dw = 4e^w$. Thus,we have $\frac{dy}{dt} = \frac{dy}{dw}\frac{dw}{dt} = 12e^{3t}$

   (c)Similarly to (b) above, $dy/dt = -12e^{-2t}$.

3. The first two derivatives are $y'(t) = y''(t) = e^t = (2.718)^t$. The value of t can be either positive, zero, or negative. If $t > 0$, then e^t is clearly positive; if $t = 0$, then $\mathrm{e}^t = 1$, again positive; finally, if $t < 0$, say $t = -2$, then $\mathrm{e}^t = 1/\left(2.718\right)^2$, still positive. Thus $y'(t)$ and $y''(t)$ are always positive, and the function $y = e^t$ always increases at an increasing rate.

4. (a) The curve with $a = -1$ is the mirror image of the curve with $a = 1$ with reference to the horizontal axis.

   (b) The curve with $c = -1$ is the mirror image of the curve with $c = 1$ with reference to the vertical axis.

Exercise 10.2

1. (a) $e^2 = 1+2+\frac{1}{2}(2)^2+\frac{1}{6}(2)^3+\frac{1}{24}(2)^4+\frac{1}{120}(2)^5+\frac{1}{720}(2)^6+\frac{1}{5040}(2)^7+\frac{1}{40320}(2)^8+\frac{1}{362880}(2)^9+\frac{1}{3628800}(2)^{10}$

   $$= 1 + 2 + 2 + 1.333 + 0.667 + 0.267 + 0.089 + 0.025 + 0.006 + 0.001 + 0.000 = 7.388$$

   (b) $e^{1/2} = 1+\frac{1}{2}+\frac{1}{2}(\frac{1}{2})^2+\frac{1}{6}(\frac{1}{2})^3+\frac{1}{24}(\frac{1}{2})^4+\frac{1}{120}(\frac{1}{2})^5 = 1+0.5+0.125+0.021+0.003+0.000 = 1.649$

2. (a) The derivatives are: $\phi' = 2e^{2x}$, $\phi'' = 2^2e^{2x}$, $\phi''' = 2^3e^{2x}$, or in general $\phi^{(k)} = 2^k e^{2x}$. Thus we have $\phi'(0) = 2$, $\phi''(0) = 2^2$, or more generally $\phi^{(k)}(0) = 2^k$. Accordingly,

   $P_n = 1 + 2x + \frac{1}{2!}2^2x^2 + \frac{1}{3!}2^3x^3 + \cdots + \frac{1}{n!}2^nx^n = 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots + \frac{1}{n!}(2x)^n$

   (b) $R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}x^{n+1} = \frac{2^{(n+1)}e^{2p}}{(n+1)!}x^{n+1} = \frac{e^{2p}}{(n+1)!}(2x)^{n+1}$

   It can be verified that $R_n \to 0$ as $n \to \infty$.

   (c) Hence $\phi(x)$ can be expressed as an infinite series:

   $\phi(x) = 1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots$

3. (a) $\$70e^{0.04(3)} = \$70e^{0.12}$ (b) $\$690e^{0.05(2)} = \$690e^{0.10}$

4. (a) 0.07 (or 7%) (b) 0.03 (c) 0.40 (d) 1 (or 100%)

5. When $t = 0$, the two functions have the same value (the same y intercept). Also, $y_1 = Ae^r$ when $t = 1$, but $y_2 = Ae^r$ when $t = -1$. Generally, $y_1 = y_2$ whenever the value of t in one function is the negative of the t value in the other; hence the mirror- image relationship.

Exercise 10.3

1. (a) 4 (b) -4 (c) 4 (d) 5

2. (a) 7 (b) -4 (c) -3 (d) -2 (e) 6 (f) 0

3. (a) $\log_{10}(100)^{13} = 13\log_{10}100 = 13(2) = 26$

   (b) $\log_{10}(\frac{1}{100}) = \log_{10}1 - \log_{10}100 = 0 - 2 = -2$

   (c) $\ln\frac{3}{B} = \ln 3 - \ln B$

   (d) $\ln\ Ae^2 = \ln A = \ln e^2 = \ln A + 2$

   (e) $\ln ABe^{-4} = \ln A + \ln B + \ln e^{-4} = \ln A + \ln B - 4$

   (f) $(\log_4 e)(\log_e 64) = \log_4 64 = 3$

4. (a) and (c) are valid; (b) and (d) are not.

5. By definition, $e^{\ln(u/v)} = \frac{u}{v}$. But we can also write $\frac{u}{v} = \frac{e^{\ln u}}{e^{\ln v}} = e^{(\ln u - \ln v)}$. Equating the two expressions for $\frac{u}{v}$, we obtain $\ln\frac{u}{v} = \ln u - \ln v$.

Exercise 10.4

1. If $r = 0$, then $y = Ae^{rt} = Ae^0 = A$, and the function degenerates into a constant function. The nonzero requirement serves to preclude this contingency.

2. The graphs are of the same general shape as in Fig. 10.3; the y intercepts will be A (i.e., $y = A$) for both.

3. Since $y = ab^{ct}$, we have $\log_b y = \log_b a + ct \log_b b = \log_b a + ct$.

   Thus, by solving for t, we get

   $t = \frac{\log_b y - \log_b a}{c} \qquad (c \neq 0)$

   This is the desired inverse function because it expresses t in terms of y.

4. (a) $a = 1$, $b = 8$, and $c = 3$; thus $r = 3\ln 8$, and $y = e^{(3\ln 8)t}$. We can also write this as $y = e^{6.2385t}$.

   (b) $a = 2$, $b = 7$, and $c = 2$; thus $r = 2\ln 7$, and $y = 2e^{(2\ln 7)t}$. We can also write this as $y = 2e^{3.8918t}$.

   (c) $a = 5$, $b = 5$, and $c = 1$; thus $r = \ln 5$, and $y = 5e^{(\ln 5)t}$. We can also write this as $y = 5e^{1.6095t}$.

   (d) $a = 2$, $b = 15$, and $c = 4$; thus $r = 4\ln 15$, and $y = 2e^{(4\ln 15)t}$. We can also write this as $y = 2e^{10.8324t}$..

5. (a) $a = 1$, $b = 7$, $c = 1$; thus $t = \frac{1}{\ln 7}\ln y (= \frac{1}{1.9459}\ln y = 0.5139 \ln y)$

   (b) $a = 1$, $b = 8$, $c = 3$; thus $t = \frac{1}{\ln 8}\ln 3y (= \frac{1}{2.0795}\ln 3y = 0.4809\ln 3y)$

   (c) $a = 3$, $b = 15$, $c = 9$; thus $t = \frac{3}{\ln 15}\ln 9y (= \frac{3}{2.7081}\ln 9y = 1.1078\ln 9y)$

   (d) $a = 2$, $b = 10$, $c = 1$; thus $t = \frac{2}{\ln 10}\ln y (= \frac{2}{2.3026}\ln y = 0.8686\ln y)$

6. The conversion involved is $Ae^{rt} = A(1 + \frac{i}{c})^{ct}$, where c represents the number of compoundings per year. Similarly to formula (10.18), we can obtain a general conversion formula $r = c\ln(1 + \frac{i}{c})$.

   (a) $c = 1$, and $i = 0.05$; thus $r = \ln 1.05$.

   (b) $c = 2$, and $i = 0.05$; thus $r = 2\ln 1.025$.

   (c) $c = 2$, and $i = 0.06$; thus $r = 2\ln 1.03$.

(d) $c = 4$, and $i = 0.06$; thus $r = 4 \ln 1.015$.

7. (a)The 45^0 line drawn through the origin serves as a mirror.

   (b)Yes. the horizontal axis is a mirror

   (c) Yes the horizontal axis is a mirror

Exercise 10.5

1. (a) $2e^{2t+4}$ (b) $-9e^{1-7t}$ (c) $2te^{t^2+1}$ (d) $-10te^{2-t^2}$ (e) $(2ax + b)e^{ax^2+bx+c}$

   (f) $\frac{dy}{dx} = x\frac{d}{dx}e^x + e^x\frac{dx}{dx} = xe^x + e^x = (x+1)e^x$

   (g) $\frac{dy}{dx} = x^2(2e^{2x}) + 2xe^{2x} = 2x(x+1)e^{2x}$

   (h) $\frac{dy}{dx} = a(xbe^{bx+c} + e^{bx+c}) = a(bx+1)e^{bx+c}$

2. (a) $\frac{d}{dt}\ln at = \frac{d}{dt}(\ln a + \ln t) = 0 + \frac{d}{dt}\ln t = \frac{1}{t}$.

   (b) $\frac{d}{dt}\ln t^c = \frac{d}{dt}c\ln t = c\frac{d}{dt}\ln t = \frac{c}{t}$.

3. (a) $\frac{dy}{dt} = \frac{35t^4}{7t^5} = \frac{5}{t}$ (b) $\frac{dy}{dt} = \frac{act^{c-1}}{at^c} = \frac{c}{t}$

   (c) $\frac{dy}{dt} = \frac{1}{t+99}$ (d) $\frac{dy}{dt} = 5\frac{2(t+1)}{(t+1)^2} = \frac{10}{t+1}$

   (e) $\frac{dy}{dx} = \frac{1}{x} - \frac{1}{1+x} = \frac{1}{x(1+x)}$

   (f) $\frac{dy}{dx} = \frac{d}{dx}[\ln x + 8\ln(1-x)] = \frac{1}{x} + \frac{-8}{1-x} = \frac{1-9x}{x(1-x)}$

   (g) $\frac{dy}{dx} = \frac{d}{dx}[\ln 2x - \ln(1+x)] = \frac{2}{2x} - \frac{1}{1+x} = \frac{1}{x(1+x)}$

   (h) $\frac{dy}{dx} = 5x^4\frac{2x}{x^2} + 20x^3\ln x^2 = 10x^3(1 + 2\ln x^2) = 10x^3(1 + 4\ln x)$

4. (a) $\frac{dy}{dt} = 5^t \ln 5$ (b) $\frac{dy}{dt} = \frac{1}{(t+1)\ln 2}$ (c) $\frac{dy}{dt} = 2(13)^{2t+3}\ln 13$

   (d) $\frac{dy}{dx} = \frac{14x}{7x^2}\frac{1}{\ln 7} = \frac{2}{x\ln 7}$ (e) $\frac{dy}{dx} = \frac{16x}{(8x^2+3)\ln 2}$

   (f) $\frac{dy}{dx} = x^2\frac{d}{dx}\log_3 x + \log_3 x\frac{d}{dx}x^2 = x^2\frac{1}{x\ln 3} + (\log_3 x)(2x) = \frac{x}{\ln 3} + 2x\log_3 x$

5. (a) Let $u = f(t)$, so that $du/dt = f'(t)$. Then

   $\frac{d}{dt}b^{f(t)} = \frac{db^u}{dt} = \frac{db^u}{du}\frac{du}{dt} = (b^u \ln b)f'(t) = f'(t)b^{f(t)}\ln b$

   (b) Let $u = f(t)$. Then

   $\frac{d}{dt}\log_b f(t) = \frac{d}{dt}\log_b u = \frac{d}{du}\log_b u\frac{du}{dt} = \frac{1}{u\ln b}f'(t) = \frac{f'(t)}{f(t)}\frac{1}{\ln b}$

6. For $V = Ae^{rt}$, the first two derivatives are $V' = rAe^{rt} > 0$ and $V'' = r^2Ae^{rt} > 0$

   Thus V is strictly increasing at an increasing rate, yielding a strictly convex curve. For $A = Ve^{-rt}$, the first two derivatives are

   $A' = -rVe^{-rt} < 0$ and $A'' = r^2Ve^{-rt} > 0$

   Thus A is strictly decreasing at an increasing rate (with the negative slope taking smaller numerical values as t increases), also yielding a strictly convex curve.

7. (a) Since $\ln y = \ln 3x - \ln(x+2) - \ln(x+4)$, we have

   $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} - \frac{1}{x+2} - \frac{1}{x+4} = \frac{8-x^2}{x(x+2)(x+4)}$ and $\frac{dy}{dx} = \frac{8-x^2}{x(x+2)(x+4)}\frac{3x}{(x+2)(x+4)} = \frac{3(8-x^2)}{(x+2)^2(x+4)^2}$

   (b) Since $\ln y = \ln(x^2+3) + x^2 + 1$, we have

   $\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+3} + 2x = \frac{2x(x^2+4)}{x^2+3}$ and $\frac{dy}{dx} = \frac{2x(x^2+4)}{x^2+3}(x^2+3)e^{x^2+1} = 2x(x^2+4)e^{x^2+1}$

Exercise 10.6

1. Since $A = Ke^{2\sqrt{t}-rt}$, we have $\ln A = \ln K + 2\sqrt{t} - rt$. Differentiation with respect to t yields

   $\frac{1}{A}\frac{dA}{dt} = t^{-1/2} - r$ or $\frac{dA}{dt} = A(t^{-1/2} - r)$

   Setting $\frac{dA}{dt} = 0$, we then find: $t^* = 1/r^2$.

   In the second derivative, $\frac{d^2A}{dt^2} = A\frac{d}{dt}(t^{-1/2} - r) + (t^{-1/2} - r)\frac{dA}{dt}$

   the second term vanishes when $\frac{dA}{dt} = 0$. Thus $\frac{d^2A}{dt^2} = -A/2\sqrt{t^3} < 0$, which satisfies the second-order condition for a maximum.

2. $\frac{d^2A}{dt^2} = A\frac{d}{dt}(\frac{\ln 2}{2\sqrt{t}} - r) + (\frac{\ln 2}{2\sqrt{t}} - r)\frac{dA}{dt} = A\frac{d}{dt}(\frac{\ln 2}{2}t^{-1/2} - r) + 0 = \frac{-A\ln 2}{4\sqrt{t^3}} < 0$ [Since $A < 0$ and $\ln 2 > 0$]

   Thus the second-order condition is satisfied.

3. (a) Since $A = Ve^{-rt} = f(t)\ e^{-rt}$, we have $\ln A = \ln f(t) - rt$, and $\frac{1}{A}\frac{dA}{dt} = \frac{f'(t)}{f(t)} - r = r_v - r$ or $\frac{dA}{dt} = A(r_v - r)$

   Inasmuch as A is nonzero, $dA/dt = 0$ if and only if $r_v = r$.

   (b) The second derivative is $\frac{d^2A}{dt^2} = A\frac{d}{dt}\frac{f'(t)}{f(t)} = A\frac{d}{dt}r_v < 0$ iff $\frac{d}{dt}r_v < 0$

4. The value of t^* depends only on the parameter r. Since $\frac{dt^*}{dr} = \frac{d}{dr}\frac{1}{4}r^{-2} = -\frac{1}{2r^3} < 0$ a higher interest rate means a smaller t^*(an earlier optimal time of sale).

Exercise 10.7

1. (a) $\ln y = \ln 5 + 2\ln t$; thus $r_y = \frac{d}{dt}\ln y = \frac{2}{t}$.

   (b) $\ln y = \ln a + c\ln t$; thus $r_y = c/t$.

   (c) $\ln y = \ln a + t\ln b$; thus $r_y = \ln b$.

   (d) Let $u = 2^t$ and $v = t^2$. Then $r_u = \ln 2$, and $r_v = 2/t$. Thus $r_y = r_u + r_v = \ln 2 + 2/t$. Alternatively, we can write $\ln y = t\ln 2 + 2\ln t$; thus

   $r_y = \frac{d}{dt}\ln y = \ln 2 = \frac{2}{t}$

   (e) Let $u = t$ and $v = 3^t$. Then $r_u = \frac{d(\ln u)}{dt} = \frac{d(\ln t)}{dt} = \frac{1}{t}$, and $r_v = \frac{d(\ln v)}{dt} = \frac{d(\ln 3^t)}{dt} = \frac{d(t\ln c)}{dt} = \ln 3$. Consequently, $r_y = r_u + r_v = \frac{1}{t} - \ln 3$.

   Alternatively, we can write $\ln y = \ln t - t\ln 3$; thus $r_y = \frac{d}{dt}\ln y = \frac{1}{t} - \ln 3$

2. $\ln H = \ln H_0 + bt\ln 2$; thus $r_H = b\ln 2$. Similarly, $\ln C = \ln C_0 + at\ln e$; thus $r_C = a\ln e = a$. It follows that $r_{(C/H)} = r_C - r_H = a - b\ln 2$.

3. Taking log, we get $\ln y = k\ln x$. Differentiating with respect to t, we then obtain $r_y = kr_x$.

4. $y = \frac{u}{v}$ implies $\ln y = \ln u - \ln v$; it follows that $r_y = \frac{d}{dt}\ln y = \frac{d}{dt}\ln u - \frac{d}{dt}\ln v = r_u - r_v$

5. By definition, $y = Y/P$. Taking the natural log, we have $\ln y = \ln Y - \ln P$. Differentiation of $\ln y$ with respect to time t yields $\frac{d}{dt}\ln y = \frac{d}{dt}\ln Y - \frac{d}{dt}\ln P$. which means $r_y = r_Y - r_P$ were r_P is the rate of inflation.

6. $z = u - v$ implies $\ln z = \ln(u - v)$; thus $r_z = \frac{d}{dt}\ln z = \frac{d}{dt}\ln(u - v) = \frac{1}{u-v}\frac{d}{dt}\ln(u - v) = \frac{1}{u-v}\frac{d}{dt}\ln[f'(t) - g'(t)] = \frac{1}{u-v}(ur_u - vr_v)$

7. $\ln Q_d = \ln k - n\ln P$. Thus, by (10.28), $\epsilon_d = -n$, and $|\epsilon_d| = n$.

8. (a) Since $\ln y = \ln w + \ln z$, we have $\epsilon_{yx} = \frac{d(\ln y)}{d(\ln x)} = \frac{d(\ln w)}{d(\ln x)} + \frac{d(\ln z)}{d(\ln x)} = \epsilon_{wx} + \epsilon_{zx}$

   (b) Since $\ln y = \ln u - \ln v$, we have $\epsilon_{yx} = \frac{d(\ln y)}{d(\ln x)} = \frac{d(\ln u)}{d(\ln x)} - \frac{d(\ln v)}{d(\ln x)} = \epsilon_{ux} + \epsilon_{vx}$

9. Let $u = \log_b y$, and $v = \log_b x$ (implying that $x = b^v$). Then $\frac{du}{dv} = \frac{du}{dy}\frac{dy}{dx}\frac{dx}{dv} = \frac{1}{y}(\log_b e)\frac{dy}{dx}b^v\ln b$

   Since $\log_b e = \frac{1}{\ln b}$, and since $b^v = x$, we have $\frac{du}{dv} = \frac{x}{y}\frac{dy}{dx} = \epsilon_{yx}$

10. Since $M_d = f[Y(t), i(t)]$, we can write the total derivative $\frac{dM_d}{dt} = f_y\frac{dy}{dt} + f_i\frac{di}{dt}$

Thus the rate of growth of M_d is $r_{M_d} = \frac{dM_d/dt}{M_d} = \frac{f_Y}{f}\frac{dY}{dt} + \frac{f_i}{f}\frac{di}{dt} = \frac{f_Y}{f}\frac{Y}{Y}\frac{dY}{dt} + \frac{f_i}{f}\frac{i}{i}\frac{di}{dt} = \frac{f_Y Y}{f}(\frac{1}{Y}\frac{dY}{dt}) + \frac{f_i i}{f}(\frac{1}{i}\frac{di}{dt}) = \epsilon_{M_d Y} r_Y + \epsilon_{M_d i} r_i$

Alternatively, using logarithms, we may write $r_{M_d} = \frac{d}{dt}\ln M_d = \frac{1}{M_d}\frac{d}{dt}M_d$, but this then leads us back to the preceding process.

11. By the same procedure used in 9 above, we can find that $r_Q = \epsilon_{QK} r_K + \epsilon_{QL} r_L$

CHAPTER 11

Exercise 11.2

1. The derivatives are: $f_x = 2x + y$, $f_y = x + 4y$, $f_{xx} = 2$, $f_{yy} = 4$, and $f_{xy} = 1$. The first-order condition requires that $2x + y = 0$ and $x + 4y = 0$. Thus we have

   $x^* = y^* = 0$ implying $z^* = 3$ (which is a minimum)

2. The derivatives are: $f_x = -2x + 6$, $f_y = -2y + 2$, $f_{xx} = -2$, $f_{yy} = -2$, and $f_{xy} = 0$. The first-order condition requires that $-2x = 6$ and $-2y = -2$. Thus we find

   $x^* = 3$ $y^* = 1$ so that $z^* = 10$ (which is a maximum)

3. $f_x = 2ax$, $f_y = 2by$, $f_{xx} = 2a$, $f_{yy} = 2b$, and $f_{xy} = 0$. The first-order condition requires that $2ax = 0$ and $2by = 0$. Thus

   $x^* = y^* = 0$ so that $z^* = c$

   The second derivatives give us $f_{xx}f_{yy} = 4ab$, and $f_{xy}^2 = 0$. Thus:

   (a) z^* is a minimum if a, $b > 0$.

   (b) z^* is a maximum if a, $b < 0$.

   (c) z^* gives a saddle point if a and b have opposite signs.

4. $f_x = 2\left(e^{2x} - 1\right)$, $f_y = 4y$, $f_{xx} = 4e^{2x}$, $f_{yy} = 4$, and $f_{xy} = 0$. The first-order condition requires that $e^{2x} = 1$ and $4y = 0$. Thus

   $x^* = y^* = 0$ so that $z^* = 4$

   Since $f_{xx}f_{yy} = 4(4)$ exceeds $f_{xy}^2 = 0$, $z^* = 4$ is a minimum.

5.

   (a) And pair (x, y) other than $(2, 3)$ yields a positive z value.

   (b) Yes. At $x^* = 2$ and $y^* = 3$, we find

   $$f_x = 4\left(x - 2\right)^3 \quad \text{and} \quad f_y = 4\left(y - 3\right)^3 = 0$$

   (c) No. At $x^* = 2$ and $y^* = 3$, we have $f_{xx} = f_{yy} = f_{xy} = f_{yx} = 0$.

   (d) By (11.6), $d^2z = 0$. Thus (11.9) is satisfied.

Exercise 11.3

1. (a) $q = 4u^2 + 4uv + 3v^2$

   (b) $q = -2u^2 + 4uv - 4v^2$

   (c) $q = 5x^2 + 6xy$

   (d) $q = f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}dy^2$

2. For (b): $q = -2u^2 + 4uv - 4v^2$. For (c): $q = 5x^2 + 6xy$. Both are the same as before.

3.

   (a) $\begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$: $4 > 0$, $4(3) > 2^2$ – positive definite

   (b) $\begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$: $-2 < 0$, $-2(-4) > 2^2$ – negative definite

   (c) $\begin{bmatrix} 5 & 3 \\ 3 & 0 \end{bmatrix}$: $5 > 0$, $5(0) < 3^2$ – neither

4.

   (a) $q = [u \quad v] \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

   (b) $q = [u \quad v] \begin{bmatrix} 1 & 3.5 \\ 3.5 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

   (c) $q = [u \quad v] \begin{bmatrix} -1 & 4 \\ 4 & -31 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

   (d) $q = [x \quad y] \begin{bmatrix} -2 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

   (e) $q = [u_1 \quad u_2 \quad u_3] \begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

   (f) $q = [u \quad v \quad w] \begin{bmatrix} -1 & 2 & -3 \\ 2 & -4 & 0 \\ -3 & 0 & -7 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$

5.

(a) $3 > 0,\ 3(7) > (-2)^2$ – positive definite

(b) $1 > 0,\ 1(3) < (3.5)^2$ – neither

(c) $-1 < 0,\ -1(-31) > 4^2$ – negative definite

(d) $-2 < 0,\ -2(-5) > 3^2$ – negative definite

(e) $3 > 0, \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix} = 14 > 0, \begin{vmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{vmatrix} = 37 > 0$ – positive definite

(f) $-1 < 0, \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} = 0$ – neither (no need to check $|D_3|$)

6.

(a) The characteristic equation is
$$\begin{vmatrix} 4-r & 2 \\ 2 & 3-r \end{vmatrix} = r^2 - 7r + 8 = 0$$
Its roots are $r_1,\ r_2 = \frac{1}{2}\left(7 + \sqrt{17}\right)$. Both roots being positive, $u'Du$ is positive definite.

(b) The characteristic equation is $r^2 + 6r + 4 = 0$, with roots $r_1,\ r_2 = -3 \pm \sqrt{5}$. Both roots being negative, $u'Eu$ is negative definite.

(c) The characteristic equation is $r^2 - 5r - 9 = 0$, with roots
$r_1,\ r_2 = \frac{1}{2}\left(5 \pm \sqrt{61}\right)$. Since r_1 is positive, but r_2 is negative, $u'Fu$ is indefinite.

7. The characteristic equation $\begin{vmatrix} 4-r & 2 \\ 2 & 1-r \end{vmatrix} = r^2 - 5r = 0$ has the roots $r_1 = 5$ and $r_2 = 0$.
(Note: This is an example where $|D| = 0$). Using r_1 in (11.13'), we have $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. Thus $x_1 = 2x_2$. Upon normalization, we obtain the first characteristic vector
$$v_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$
Next, using r_2 in (11.13'), we have $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$. Therefore, $x_1 = -\frac{1}{2}x_2$. Upon normalization, we obtain

$$v_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

These results happen to be identical with those in Example 5.

8. The characteristic equation can be written as

$$r^2 - (d_{11} + d_{22})\, r + (d_{11}d_{22} - d_{12}d_{21}) = 0$$

Thus $r_1,\, r_2 = \frac{1}{2}\left[(d_{11} + d_{22}) \pm \sqrt{(d_{11} + d_{22})^2 - 4\,(d_{11}d_{22} - d_{12}d_{21})}\right]$

(a) The expression under the square-root sign can be written as

$$E = d_{11}^2 + 2d_{11}d_{22} + d_{22}^2 - 4d_{11}d_{22} + 4d_{12}d_{21}$$
$$= d_{11}^2 - 2d_{11}d_{22} + d_{22}^2 + 4d_{12}d_{21} = (d_{11} - d_{22})^2 + 4d_{12}^2 \geq 0$$

Thus no imaginary number can occur in r_1 and r_2.

(b) To have repeated roots, E has to be zero, which can occur if and only if $d_{11} = d_{22}$ (say, =c) and at the same time $d_{12} = d_{21} = 0$. This would mean that matrix D takes the form of $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$.

(c) Positive or negative semidefiniteness allows a characteristic root to be zero (r=0), which implies the possibility that the characteristic equation reduces to $d_{11}d_{22} - d_{12}d_{21} = 0$, or $|D| = 0$.

Exercise 11.4

1. The first-order condition

$$f_1 = 2x_1 - 3x_2 = 0$$
$$f_2 = -3x_1 + 6x_2 + 4x_3 = 0$$
$$f_3 = 4x_2 + 12x_3 = 0$$

is a homogeneous linear-equation system in which the three equations are independent. Thus the only solution is

$$x_1^* = x_2^* = x_3^* = 0 \qquad \text{so that} \qquad z^* = 0$$

The Hessian is $\begin{vmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{vmatrix}$, with $|H_1| = 2 > 0$, $|H_2| = 3 > 0$, and $|H_3| = 4 > 0$.

Consequently, $z^* = 0$ is a minimum.

2. The first-order condition consists of the three equations

   $$f_1 = -2x_1 = 0 \qquad f_2 = -2x_2 = 0 \qquad f_3 = -2x_3 = 0$$

   Thus $\quad x_1^* = x_2^* = x_3^* = 0 \quad$ so that $\quad z^* = 29$

   The Hessian is $\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix}$, with $|H_1| = -2 < 0$, $|H_2| = 4 > 0$, and $|H_3| = -8 < 0$.
   Consequently, $z^* = 29$ is a maximum.

3. The three equations in the first-order conditions are

   $$2x_1 + x_3 = 0$$

   $$2x_2 + x_3 = 1$$

   $$x_1 + x_2 + 6x_3 = 0$$

   Thus $\quad x_1^* = \frac{1}{20} \quad x_2^* = \frac{11}{20} \quad x_3^* = -\frac{2}{20} \quad$ so that $\quad z^* = -\frac{11}{40}$

   Since the Hessian is $\begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix}$, with $|H_1| = 2 > 0$, $|H_2| = 4 > 0$, and $|H_3| = 20 > 0$, the z^* value is a minimum.

4. By the first-order condition, we have

   $$f_x = 2e^{2x} - 2 = 0, \qquad f_y = -e^{-y} + 1 = 0, \qquad f_w = 2we^{w^2} - 2e^w = 0$$

   Thus $\quad x^* = 0 \quad y^* = 0 \quad \bar{w} = 1 \quad$ so that $\quad z^* = 2 - e$

   Note:The values of x^* and y^* are found from the fact that $e^0 = 1$. Finding w^* is more complicated. One way of doing it is as follows: First, rewrite the equation $f_w = 0$ as

   $$we^{w^2} = e^w$$

   Taking natural logs yield

   or

   $$\ln w + \ln e^{w^2} = \ln e^w$$
   $$\text{or} \quad \ln w + w^2 = w$$
   $$\text{or} \quad \ln w = w - w^2$$

If we draw a curve for $\ln w$, and another for $w - w^2$, their intersection point will give us the solution. The $\ln w$ curve is a strictly concave curve with horizontal intercept at $w = 1$. The $w - w^2$ is a hill-type parabola with horizontal intercepts $w = 0$ and $w = 1$. Thus the solution is $w^* = 1$.

The Hessian is $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4e \end{vmatrix}$ when evaluated at the stationary point, with all leading principal minors positive. Thus z^* is a minimum.

5.

(a) Problems 2 and 4 yield diagonal Hessian matrices. The diagonal elements are all negative for problem 2, and all positive for problems 4 and 5.

(b) According to (11.16), these diagonal elements represent the characteristic roots. Thus the characteristic roots are all negative (d^2z negative definite) for problem 2, and all positive (d^2z positive definite) for problem 4.

(c) Yes.

6.

(a) The characteristic equation is, by (11.14):

$$\begin{vmatrix} 2-r & 0 & 1 \\ 0 & 2-r & 1 \\ 1 & 1 & 6-r \end{vmatrix} = 0$$

Expanding the determinant by the method of Fig. 5.1, we get

$$(2-r)(2-r)(6-r) - (2-r) - (2-r) = 0$$

or $(2-r)[(2-r)(6-r) - 2] = 0$ [factoring]

or $(2-r)\left(r^2 - 8r + 10\right) = 0$

Thus, from the $(2-r)$ term, we have $r_1 = 2$. By the quadratic formula, we get from the other term: $r_2, r_3 = 4 \pm \sqrt{6}$.

(b) All three roots are positive. Thus d^2z is positive definite, and z^* is a minimum.

(c) Yes.

Exercise 11.5

1.

(a) Let u and v be any two distinct points in the domain. Then

$$f(u) = u^2 \qquad f(v) = v^2 \qquad f[\theta u + (1-\theta)v] = [\theta u + (1-\theta)v]^2$$

Substituting these into (11.20), we find the difference between the left- and right-side expressions in (11.20) to be

$$\theta u^2 + (1-\theta)v^2 - \theta^2 u^2 - 2\theta(1-\theta)uv - (1-\theta)^2 v^2$$
$$= (1-\theta)u^2 - 2\theta(1-\theta)uv + \theta(1-\theta)v^2$$
$$= (1-\theta)(u-v)^2 > 0 \qquad [\text{since } u \neq v\ ]$$

Thus $z = x^2$ is a strictly convex function.

(b) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two distinct points in the domain. Then

$$f(u) = u_1^2 + 2u_2^2 \qquad f(v) = v_1^2 + 2v_2^2$$
$$f[\theta u + (1-\theta)v] = [\theta u_1 + (1-\theta)v_1]^2 + 2[\theta u_2 + (1-\theta)v_2]^2$$

The difference between the left- and right-side expressions in (11.20) is

$\theta(1-\theta)\left(u_1^2 - 2u_1v_1 + v_1^2 + 2u_2^2 - 4u_2v_2 + 2v_2^2\right) = \theta(1-\theta)\left[(u_1-v_1)^2 + 2(u_2-v_2)^2\right] >$
0

Thus $z = x_1^2 + 2x_2^2$ is a strictly convex function.

(c) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two distinct points in the domain. Then

$$\begin{aligned} f(u) &= 2u_1^2 - u_1u_2 + u_2 \qquad fv = 2v_1^2 - v_1v_2 + v_2^2 \\ f[\theta u + (1-\theta)v] &= 2\left[\theta u_1 + (1-\theta)v_1^2\right] - [\theta u_1 + (1-\theta)v_1] \cdot [\theta u_2 + (1-\theta)v_2] \\ &\quad + [\theta u_2 + (1-\theta)v_2]^2 \end{aligned}$$

The difference between the left- and right-side expressions in (11.20) is

$\theta(1-\theta)\left[\left(2u_1^2 - 4u_1v_1 + 2v_1^2\right) - u_1u_2 + u_1v_2 + v_1u_2 - v_1v_2 + \left(u_2^2 - 2u_2v_2 + v_2^2\right)\right]$
$= \theta(1-\theta)\left[2(u_1-v_1)^2 - (u_1-v_1)(u_2-v_2) + (u_2-v_2)^2\right] > 0$

because the bracketed expression is positive, like $\theta(1-\theta)$. [The bracketed expression, a positive-definite quadratic form in the two variables $(u_1 - v_1)$ and $(u_2 - v_2)$, is positive since $(u_1 - v_1)$ and $(u_2 - v_2)$ are not both zero in our problem.] Thus $z = 2x^2 - xy + y^2$ is a strictly convex function.

2.

(a) With $f'(u) = -2u$, the difference between the left- and right-side expressions in (11.24) is

$$-v^2 + u^2 + 2u(v-u) = -v^2 + 2uv - u^2 = -(v-u)^2 < 0$$

Thus $z = -x^2$ is strictly concave.

(b) Since $f_1(u_1, u_2) = f_2(u_1, u_2) = 2(u_1 + u_2)$, the difference between the left- and right-side expressions in (11.24') is

$$(v_1 + v_2)^2 - (u_1 + u_2)^2 - 2(u_1 + u_2)[(v_1 - u_1) + (v_2 - u_2)]$$
$$= (v_1 + v_2)^2 - 2(v_1 + v_2)(u_1 + u_2) + (u_1 + u_2)^2$$
$$= [(v_1 + v_2) - (u_1 + u_2)]^2 \geq 0$$

A zero value cannot be ruled out because the two points may be, e.g., $(u_1, u_2) = (5, 3)$ and $(v_1, v_2) = (2, 6)$. Thus $z = (x_1 + x_2)^2$ is convex, but not strictly so.

(c) Since $f_1(u_1, u_2) = -u_2$, and $f_2(u_1, u_2) = -u_1$, the difference between the left- and right-side expressions in (11.24') is

$$-v_1v_2 + u_1u_2 + u_2(v_1 - u_1) + u_1(v_2 - u_2) = -v_1v_2 + v_1u_2 + u_1v_2 - u_1u_2 = (v_1 - u_1)(v_2 - u_2) \gtreqless 0$$

Thus $z = -xy$ is neither convex nor concave.

3. No. That theorem gives a sufficient condition which is not satisfied.

4.

(a) No.

(b) No.

(c) Yes.

5.

(a) The circle with its interior, i.e. a disk.

(b) Yes.

6.

(a) The set of points on an exponential curve; not a convex set.

(b) The set of points lying on or above an exponential curve; a convex set.

(c) The set of points lying on or below an inverse U-shaped curve; a convex set.

(d) The set of points lying on or above a rectangular hyperbola in the positive quadrant; a convex set.

7.

(a) This is a convex combination, with $\theta = 0.5$.

(b) This is again a convex combination, with $\theta = 0.2$.

(c) This is not a convex combination.

8.

(a) This set is the entire 2-space.

(b) This set is a cone bounded on one side by a ray passing through point u, and on the other side by a ray passing through point v.

(c) This set is the line segment uv.

9.

(a) $S^{\leq} \equiv \{(x_1, ..., x_n) \mid f(x_1, ..., x_n) \leq k\}$ (f convex)
$S^{\geq} \equiv \{(x_1, ..., x_n) \mid g(x_1, ..., x_n) \geq k\}$ (g concave)

(b) $S^{\leq}$ is a solid circle (or disk); $S^{\geq}$ is a solid square.

Exercise 11.6

1.

(a) No, because the marginal cost of one commodity will be independent of the output of the other.

(b) The first-order condition is

$$\pi_1 = P_{10} - 4Q_1 = 0 \qquad \pi_2 = P_{20} - 4Q_2 = 0$$

Thus $Q_1^* = \frac{1}{4}P_{10}$ and $Q_2^* = \frac{1}{4}P_{20}$. The profit is maximized, because the Hessian is $\begin{vmatrix} -4 & 0 \\ 0 & -4 \end{vmatrix}$, with $|H_1| < 0$ and $|H_2| > 0$. The signs of the principal minors do not depend on where they are evaluated. Thus the maximum in this problem is a unique absolute maximum.

(c) $\pi_{12} = 0$ implies that the profit-maximizing output level of one commodity is independent of the output of the other (see first-order condition). The firm can operate as if it has two plants, each optimizing the output of a different product.

2.

(a) By the procedure used in Example 2 (taking Q_1 and Q_2 as choice variables), we can find

$$Q_1^* = 3\tfrac{4}{7} \qquad Q_2^* = 4\tfrac{9}{14} \qquad P_1^* = 6\tfrac{1}{14} \qquad P_2^* = 24\tfrac{2}{7}$$

(b) The Hessian is $\begin{vmatrix} -4 & 2 \\ 2 & -8 \end{vmatrix}$, with $|H_1| = -4$ and $|H_2| = 28$. Thus the sufficient condition for a maximum is met.

(c) Substituting the P^*'s and Q^*'s into the R and C functions, we get

$$R^* = 134\tfrac{43}{98} \qquad C^* = 65\tfrac{85}{98} \qquad \text{and} \qquad \bar{r} = 68\tfrac{4}{7}$$

3. $|\epsilon_{d1}| = \left|\frac{dQ_1}{dP_1}\frac{P_1^*}{Q_1^*}\right| = \frac{1}{4}\frac{39}{6} = \frac{13}{8}$. Similarly, $|\epsilon_{d2}| = \frac{1}{5}\frac{60}{9} = \frac{4}{3}$, and $|\epsilon_{d3}| = \frac{1}{6}\frac{45}{5} = \frac{3}{2}$. The highest is $|\epsilon_{d1}|$; the lowest is $|\epsilon_{d2}|$.

(a) $C' = 15 + 2Q = 15 + 2Q_1 + 2Q_2 + 2Q_3$

(b) Equating each MR to the MC, we obtain the three equations:

$$10Q_1 + 2Q_2 + 2Q_3 = 48, \qquad 2Q_1 + 12Q_2 + 2Q_3 = 90 \qquad \text{and} \qquad 2Q_1 + 2Q_2 + 14Q_3 = 60$$

Thus $Q_1^* = 2\frac{88}{97}$, $Q_2^* = 6\frac{51}{97}$, $Q_3^* = 2\frac{91}{97}$.

(c) Substituting the above into the demand equations, we get

$$P_1^* = 51\frac{36}{97}, \qquad P_2^* = 72\frac{36}{97}, \qquad P_3^* = 57\frac{36}{97}$$

(d) Since $R_1'' = -8$, $R_2'' = -10$, $R_3'' = -12$, and $C'' = 2$, we do find that:
(1) $R_1'' - C'' = -10$ (2) $R_1''R_2'' - (R_1'' + R_2'')\,C'' = 80 + 36 = 116 > 0$, and (3) $|H| = -960 - (80 + 96 + 120)\,(2) = -1552 < 0$.

4.

(a) $\pi = P_0 Q(a,b)\left(1 + \frac{1}{2}i_0\right)^{-2} - P_{a0}a - P_{b0}b$

(b) $\pi = P_0 Q(a,b)\left(1 + \frac{1}{4}i_0\right)^{-3} - P_{a0}a - P_{b0}b$

5. $Q(a,b) = 260$

Exercise 11.7

1.

(a) We may take (11.49) as the point of departure. Letting P_{a0} alone vary (i.e., letting $dP_0 = dP_{b0} = dr = dt = 0$), and dividing through by $dP_{a0} \neq 0$, we get the matrix equation

$$\begin{bmatrix} P_0 Q_{aa} e^{-rt} & P_0 Q_{ab} e^{-rt} \\ P_0 Q_{ab} e^{-rt} & P_0 Q_{bb} e^{-rt} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial a^*}{\partial P_{a0}}\right) \\ \left(\frac{\partial b^*}{\partial P_{a0}}\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence, by Cramer's Rule,

$$\left(\frac{\partial a^*}{\partial P_{a0}}\right) = \frac{P_0 Q_{bb} e^{-rt}}{|J|} < 0 \qquad \text{and} \qquad \left(\frac{\partial b^*}{\partial P_{a0}}\right) = -\frac{P_0 Q_{ab} e^{-rt}}{|J|} < 0$$

The higher the price of input a, the smaller will be the equilibrium levels of inputs a and b.

(b) Next, letting P_{b0} alone vary in (11.49), and dividing through by $dP_{b0} \neq 0$, we can obtain results similar to (a) above:

$$\left(\frac{\partial a^*}{\partial P_{b0}}\right) = -\frac{P_0 Q_{ab} e^{-rt}}{|J|} < 0 \qquad \text{and} \qquad \left(\frac{\partial b^*}{\partial P_{b0}}\right) = \frac{P_0 Q_{aa} e^{-rt}}{|J|} < 0$$

2.

(a) P_0, i_0, P_{a0}, P_{b0}.

(b) From the first-order condition, we can check the Jacobian

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial a} & \frac{\partial F^1}{\partial b} \\ \frac{\partial F^2}{\partial a} & \frac{\partial F^2}{\partial b} \end{vmatrix} = \begin{vmatrix} P_0 Q_{aa} (1+i_0)^{-1} & P_0 Q_{ab} (1+i_0)^{-1} \\ P_0 Q_{ab} (1+i_0)^{-1} & P_0 Q_{bb} (1+i_0)^{-1} \end{vmatrix}$$
$$= P_0^2 (1+i_0)^{-2} \begin{vmatrix} Q_{aa} & Q_{ab} \\ Q_{ab} & Q_{bb} \end{vmatrix}$$

the be positive at the initial equilibrium (optimum) since the second-order sufficient condition is assumed to be satisfied. By the implicit-function theorem, we can then write

$$a^* = a^*(P_0, i_0, P_{a0}, P_{b0}) \qquad \text{and} \qquad b^* = b^*(P_0, i_0, P_{a0}, P_{b0})$$

we can also write the identities

$$P_0 Q_a(a^*, b^*)(1+i_0)^{-1} - P_{a0} \equiv 0$$
$$P_0 Q_a(a^*, b^*)(1+i_0)^{-1} - P_{b0} \equiv 0$$

Taking the total differentials, we get (after rearrangement) the following pair of equations corresponding to (11.49):

$$P_0 Q_{aa} (1+i_0)^{-1} da^* + P_0 Q_{ab} (1+i_0)^{-1} db^* = -Q_a (1+i_0)^{-1} dP_0 + P_0 Q_a (1+i_0)^{-2} di_0 + dP_{a0}$$

$$P_0 Q_{ab} (1+i_0)^{-1} da^* + P_0 Q_{bb} (1+i_0)^{-1} db^* = -Q_b (1+i_0)^{-1} dP_0 + P_0 Q_b (1+i_0)^{-2} di_0 + dP_{b0}$$

Letting P_0 alone vary (i.e., letting $di_0 = dP_{a0} = dP_{b0} = 0$), and dividing through by $dP_0 \neq 0$, we get

$$\begin{bmatrix} P_0 Q_{aa} (1+i_0)^{-1} & P_0 Q_{ab} (1+i_0)^{-1} \\ P_0 Q_{ab} (1+i_0)^{-1} & P_0 Q_{bb} (1+i_0)^{-1} \end{bmatrix} \begin{bmatrix} \frac{\partial a^*}{\partial P_0} \\ \frac{\partial b^*}{\partial P_0} \end{bmatrix} = \begin{bmatrix} -Q_a (1+i_0)^{-1} \\ -Q_b (1+i_0)^{-1} \end{bmatrix}$$

Thus $\left(\frac{\partial a^*}{\partial P_0}\right) = \frac{(Q_b Q_{ab} - Q_a Q_{bb}) P_0 (1+i_0)^{-2}}{|J|} > 0$

$\left(\frac{\partial b^*}{\partial P_0}\right) = \frac{(Q_a Q_{ab} - Q_b Q_{aa}) P_0 (1+i_0)^{-2}}{|J|} > 0$

(c) Letting i_0 alone vary, we can similarly obtain

$$\begin{bmatrix} P_0 Q_{aa} (1+i_0)^{-1} & P_0 Q_{ab} (1+i_0)^{-1} \\ P_0 Q_{ab} (1+i_0)^{-1} & P_0 Q_{bb} (1+i_0)^{-1} \end{bmatrix} \begin{bmatrix} \left(\frac{\partial a^*}{\partial i_0}\right) \\ \left(\frac{\partial b^*}{\partial i_0}\right) \end{bmatrix} = \begin{bmatrix} P_0 Q_a (1+i_0)^{-2} \\ P_0 Q_b (1+i_0)^{-2} \end{bmatrix}$$

Thus, $\left(\frac{\partial a^*}{\partial i_0}\right) = \frac{(Q_a Q_{bb} - Q_b Q_{ab}) P_0^2 (1+i_0)^{-3}}{|J|} < 0$

$\left(\frac{\partial b^*}{\partial i_0}\right) = \frac{(Q_b Q_{aa} - Q_a Q_{ab}) P_0 (1+i_0)^{-3}}{|J|} < 0$

3. Differentiating (11.49) totally with respect to P_0, we get

$$Q_a e^{-rt} + P_0 Q_{aa}\left(\frac{\partial a^*}{\partial P_0}\right)e^{-rt} + P_0 Q_{ab}\left(\frac{db^*}{dP_0}\right)e^{-rt} = 0$$
$$Q_b e^{-rt} + P_0 Q_{ab}\left(\frac{\partial a^*}{\partial P_0}\right)e^{-rt} + P_0 Q_{bb}\left(\frac{db^*}{dP_0}\right)e^{-rt} = 0$$

Or, in matrix notation,

$$\begin{bmatrix} P_0 Q_{aa} e^{-rt} & P_0 Q_{ab} e^{-rt} \\ P_0 Q_{ab} e^{-rt} & P_0 Q_{bb} e^{-rt} \end{bmatrix} \begin{bmatrix} (\partial a^*/\partial P_0) \\ (\partial b^*/\partial P_0) \end{bmatrix} = \begin{bmatrix} -Q_a e^{-rt} \\ -Q_b e^{-rt} \end{bmatrix}$$

which leads directly to the results in (11.50).

4. In (11.46), the elements of the Jacobian determinant are first-order partial derivatives of the components of the first-order condition shown in (11.45). Thus, those elements are really the second-order partial derivatives of the (primitive) objective function – exactly what are used to construct the Hessian determinant.

CHAPTER 12

Exercise 12.2

1.

(a) $Z = xy + \lambda(2 - x - 2y)$. The necessary condition is:

$Z_\lambda = 2 - x - 2y = 0 \qquad Z_x = y - \lambda = 0 \qquad Z_y = x - 2\lambda = 0$

Thus $\lambda^* = \frac{1}{2}$, $x^* = 1$, $y^* = \frac{1}{2}$ – yielding $z^* = \frac{1}{2}$.

(b) $Z = xy + 4x + \lambda(8 - x - y)$. The necessary condition is:

$Z_\lambda = 8 - x - y = 0 \qquad Z_x = y + 4 - \lambda = 0 \qquad Z_y = x - \lambda = 0$

Thus $\lambda^* = 6$, $x^* = 6$, $y^* = 2$ – yielding $z^* = 36$.

(c) $Z = x - 3y - xy + \lambda(6 - x - y)$. The necessary condition is:

$Z_\lambda = 6 - x - y = 0 \qquad Z_x = 1 - y - \lambda = 0 \qquad Z_y = -3 - x - \lambda = 0$

Thus $\lambda^* = -4$, $x^* = 1$, $y^* = 5$ – yielding $z^* = -19$.

(d) $Z = 7 - y + x^2 + \lambda(-x - y)$. The necessary condition is:

$Z_\lambda = -x - y = 0 \qquad Z_x = 2x - \lambda = 0 \qquad Z_y = -1 - \lambda = 0$

Thus $\lambda^* = -1$, $x^* = -\frac{1}{2}$, $y^* = \frac{1}{2}$ – yielding $z^* = 6\frac{3}{4}$.

2.

(a) Increase; at the rate $\frac{dz^*}{dc} = \lambda^* = \frac{1}{2}$.

(b) Increase; $\frac{dz^*}{dc} = 6$.

(c) Decrease; $\frac{dz^*}{dc} = -4$

(d) Decrease; $\frac{dz^*}{dc} = -1$

3.

(a) $Z = x + 2y + 3w + xy - yw + \lambda(10 - x - y - 2w)$. Hence:

$Z_\lambda = 10 - x - y - 2w = 0 \qquad Z_x = 1 + y - \lambda = 0$

$Z_y = 2 + x - w - \lambda = 0 \qquad Z_w = 3 - y - 2\lambda = 0$

(b) $Z = x^2 + 2xy + yw^2 + \lambda\left(24 - 2x - y - w^2\right) + \upsilon\,(8 - x - w)$. Thus

$$Z_\lambda = 24 - 2x - y - w^2 = 0 \qquad Z_\upsilon = 8 - x - w = 0$$

$$Z_x = 2x + 2y - 2\lambda - \upsilon = 0 \qquad Z_y = 2x + w^2 - \lambda = 0$$

$$Z_w = 2yw - 2\lambda w - \upsilon = 0$$

4. $Z = f(x,y) + \lambda\,[0 - G(x,y)] = f(x,y) - \lambda G(x,y)$. The first-order condition becomes:

$$Z_\lambda = -G(x,y) = 0 \qquad Z_x = f_x - \lambda G_x = 0 \qquad Z_y = f_y - \lambda G_y = 0$$

5. Since the constraint $g = c$ is to prevail at all times in this constrained optimization problem, the equation takes on the sense of an identity, and it follows that dg must be zero. Then it follows that d^2g must be zero, too. In contrast, the equation $dz = 0$ is in the nature of a first-order condition – dz is not identically zero, but is being set equal to zero to locate the critical values of the choice variables. Thus d^2z does not have to be zero as a matter of course.

6. No, the sign of λ^* will be changed. The new λ^* is the negative of the old λ^*.

Exercise 12.3

1.

(a) Since $|H| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 4$, $z^* = \frac{1}{2}$ is a maximum.

(b) Since $|H| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$, $z^* = 36$ is a maximum.

(c) Since $|H| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix} = -2$, $z^* = -19$ is a minimum.

(d) Since $|H| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -2$, $z^* = 6\frac{3}{4}$ is a minimum.

2. $|H_1| = \begin{vmatrix} 0 & g_1 \\ g_1 & Z_{11} \end{vmatrix} = -g_1^2 < 0$

3. The zero can be made the last (instead of the first) element in the principal diagonal, with g_1, g_2 and g_3 (in that order appearing in the last column and in the last row.

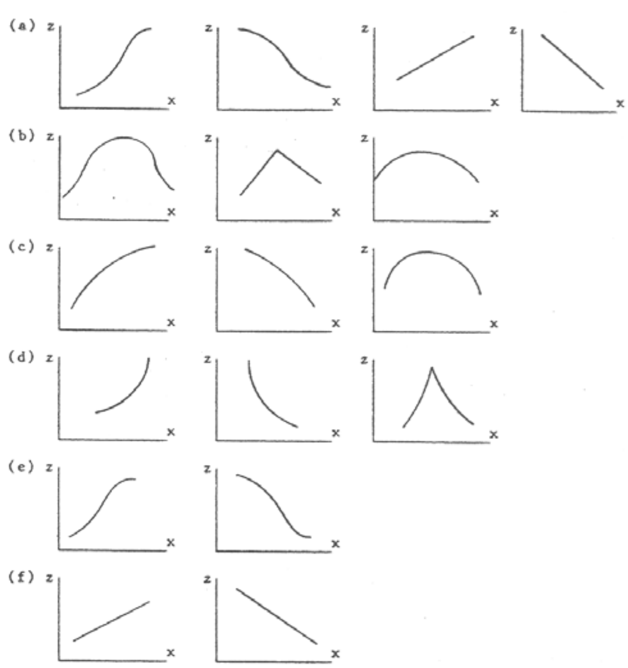
4. $$|\bar{H}| = \begin{vmatrix} 0 & 0 & g_1^1 & g_2^1 & g_3^1 & g_4^1 \\ 0 & 0 & g_1^2 & g_2^2 & g_3^2 & g_4^2 \\ g_1^1 & g_1^2 & Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ g_2^1 & g_2^2 & Z_{21} & Z_{22} & Z_{23} & Z_{24} \\ g_3^1 & g_3^2 & Z_{31} & Z_{32} & Z_{33} & Z_{34} \\ g_4^1 & g_4^2 & Z_{41} & Z_{42} & Z_{43} & Z_{44} \end{vmatrix}$$

   A sufficient condition for maximum z is $|\bar{H}_3| < 0$ and $|\bar{H}_4| = |\bar{H}| > 0$.

   A sufficient condition for minimum z is $|\bar{H}_3| > 0$ and $|\bar{H}| > 0$.

Exercise 12.4

1. Examples of acceptable curves are:



2.

(a) Quasiconcave, but not strictly so. This is because $f(v) = f(u) = a$, and thus $f[\theta u + (1-\theta)v] = a$, which is equal to (not greater than) $f(u)$.

(b) Quasiconcave, and strictly so. In the present case, $f(v) \geq f(u)$ means that $a+bv \geq a+bu$, or $v \geq u$. Moreover, to have u and v distinct, we must actually have $v > u$. Since

$$\begin{aligned} f[\theta u + (1-\theta)v] &= a + b[\theta u + (1-\theta)v] \\ &= a + b[\theta u + (1-\theta)v] + (bu - bu) \\ &= a + bu + b(1-\theta)(v-u) \\ &= f(u)\, b(1-\theta)(v-u) = f(u) + \text{some positive term} \end{aligned}$$

it follows that $f[\theta u + (1-\theta)v] > f(u)$. Hence $f(x) = a + bx$, $(b > 0)$, strictly quasi-

concave

.

(c) Quasiconcave, and strictly so. Here, $f(v) \geq f(u)$ means $a + cv^2 \geq a + cu^2$, or $v^2 \leq u^2$ (since $c < 0$). For nonnegative distinct values of u and v, this in turn means $v < u$. Now we have

$$\begin{aligned} f[\theta u + (1-\theta)v] &= a + c[\theta u + (1-\theta)v]^2 + \left(cu^2 - cu^2\right) \\ &= a + cu^2 + c\left\{[\theta u + (1-\theta)v]^2 - u^2\right\} \end{aligned}$$

Using the identity $y^2 - x^2 \equiv (y+x)(y-x)$, we can rewrite the above expression as

$a + cu^2 + c[\theta u + (1-\theta)v + u][\theta u + (1-\theta)v - u]$

$= f(u) + c[(1+\theta)u + (1-\theta)v][(1-\theta)(v-u)]$

$= f(u) +$ some positive term $> f(u)$

Hence $f(x) = a + cx^2$, $(c < 0)$, is strictly quasiconcave.

3. Both $f(x)$ and $g(x)$ are monotonic, and thus quasiconcave. However, $f(x) + g(x)$ displays both a hill and a valley. If we pick $k = 5\frac{1}{2}$, for instance, neither $S^{\geq}$ nor $S^{\leq}$ will be a convex set. Therefore $f(x) + g(x)$ is not quasiconcave.

   (a) This cubic function has a graph similar to Fig. 2.8c, with a hill in the second quadrant and valley in the fourth. If we pick $k = 0$, neither $S^{\geq}$ nor $S^{\leq}$ is a convex set. The function is neither quasiconcave nor quasiconvex.

   (b) This function is linear, and hence both quasiconcave and quasiconvex.

   (c) Setting $x_2 - \ln x_1 = k$, and solving for x_2, we get the isovalue equation $x_2 = \ln x_1 + k$. In the x_1x_2 plane, this plots for each value of k as a log curve shifted upward vertically by the amount of k. The set $S^{\leq} = \{(x_1, x_2) \mid f(x_1, x_2) \leq k\}$ – the set of points on or below the isovalue curve – is a convex set. Thus the function is quasiconvex. (but not quasiconcave).

4.

   (a) A cubic curve contains two bends, and would thus violate both parts of (12.21).

   (b) From the discussion of the cubic total-cost function in Sec. 9.4, we know that if a, c, $d > 0$, $b < 0$, and $b^2 < 3ac$, then the cubic function will be upward-sloping for nonnegative x. Then, by (12.21), it is both quasiconcave and quasiconvex.

5. Let u and v be two values of x, and let $f(v) = v^2 \geq f(u) = u^2$, which implies $v \geq u$. Since $f'(x) = 2x$, we find that

$f'(u)(v-u) = 2u(v-u) \geq 0$

$f'(v)(v-u) = 2v(v-u) \geq 0$

Thus, by (12.22), the function is both quasiconcave and quasiconvex., confirming the conclusion in Example 1.

6. The set $S^{\leq}$, involving the inequality $xy \leq k$, consists of the points lying on or below a rectangular hyperbola – not a convex set. Hence the function is quasiconvex by (12.21). Alternatively, since $f_x = y$, $f_y = x$, $f_{xx} = 0$, $f_{xy} = 1$, and $f_{yy} = 0$, we have $|B_1| = -y^2 \leq 0$ and $|B_2| = 2xy \geq 0$, which violates the necessary condition (12.25') for quasiconvexity.

7.

(a) Since $f_x = -2x$, $f_y = -2y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -2$, we have

$|B_1| = -4x^2 < 0 \qquad |B_2| = 8\left(x^2 + y^2\right) > 0$

By (12.26), the function is quasiconcave.

(b) Since $f_x = -2(x+1)$, $f_y = -2(y+2)$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -2$, we have

$|B_1| = -4(x+1)^2 < 0 \qquad |B_2| = 8(x+1)^2 + 8(y+2)^2 > 0$

By (12.26), the function is quasiconcave.

Exercise 12.5

1.

(a) $Z = (x+2)(y+1) + \lambda(130 - 4x - 6y)$

(b) The first-order condition requires that

$Z_\lambda = 130 - 4x - 6y = 0, \qquad Z_x = y + 1 - 4\lambda = 0, \qquad Z_y = x + 2 - 6\lambda = 0$

Thus we have $\lambda^* = 3$, $x^* = 16$, and $y^* = 11$.

(c) $\left|\bar{H}\right| = \begin{vmatrix} 0 & 4 & 6 \\ 4 & 0 & 1 \\ 6 & 1 & 0 \end{vmatrix} = 48 > 0.$ Hence utility is maximized.

(d) No.

2.

(a) $Z = (x+2)(y+1) + \lambda(B - xP_x - yP_y)$

(b) As the necessary condition for extremum, we have

$Z_\lambda = B - xP_x - yP_y = 0$ or $-P_x x - P_y y = -B$

$Z_x = y + 1 - \lambda P_x = 0$ $\quad - P_x\lambda + y = -1$

$Z_y = x + 2 - \lambda P_y = 0$ $\quad - P_y\lambda + x = -2$

By Cramer's Rule, we can find that

$\lambda^* = \frac{B+2P_x+P_y}{2P_xP_y}$ $\quad x^* = \frac{B-2P_x+P_y}{2P_x}$ $\quad y^* = \frac{B+2P_x-P_y}{2P_y}$

(c) $\left|\bar{H}\right| = \begin{vmatrix} 0 & P_x & P_y \\ P_x & 0 & 1 \\ P_y & 1 & 0 \end{vmatrix} = 2P_xP_y > 0.$ Utility is maximized.

(d) When $P_x = 4$, $P_y = 6$, and $B = 130$, we get $\lambda^* = 3$, $x^* = 16$ and $y^* = 11$. These check with the preceding problem.

3. Yes. $\left(\frac{\partial x^*}{\partial B}\right) = \frac{1}{2P_x} > 0$, $\left(\frac{\partial x^*}{\partial P_x}\right) = -\frac{B+P_y}{2P_x^2} < 0$, $\left(\frac{\partial x^*}{\partial P_y}\right) = \frac{1}{2P_x} > 0$, $\left(\frac{\partial y^*}{\partial B}\right) = \frac{1}{2P_y} > 0$, $\left(\frac{\partial y^*}{\partial P_x}\right) = \frac{1}{P_y} > 0$, $\left(\frac{\partial y^*}{\partial P_y}\right) = -\frac{B+2P_x}{2P_y^2} < 0$.

An increase in income B raises the level of optimal purchases of x and y both; an increase in the price of one commodity reduces the optimal purchase of that commodity itself, but raises the optimal purchase of the other commodity.

4. We have $U_{xx^*} = U_{yy} = 0$, $U_{xy} = U_{yx^*} = 1$, $|J| = \left|\bar{H}\right| = 2P_xP_y$.

$x^* = \frac{(B-2P_x+P_y)}{2P_x}$, and $\lambda^* = \frac{(B+2P_x+P_y)}{2P_xP_y}$. Thus:

(a) $\left(\frac{\partial x^*}{\partial B}\right) = \frac{1}{2P_x}$, and $\left(\frac{\partial y^*}{\partial B}\right) = \frac{1}{2P_y}$.

(b) $\left(\frac{\partial x^*}{\partial P_x}\right) = -\frac{(B+P_y)}{2P_x^2}$, and $\left(\frac{\partial y^*}{\partial P_x}\right) = \frac{1}{P_y}$.

These answers check with the preceding problem.

5. A negative sign for that derivative can mean either that the income effect (T_1) and the substitution effect (T_2) in (12.33') are both negative (normal good), or that the income effect is positive (inferior good) but is overshadowed by the negative substitution effect. The statement is not valid.

6. The optimal utility level can be expressed as $U^* = U^*(x^*, y^*)$. Thus $dU^* = U_x\, dx^* + U_y\, dy^*$, where U_x and U_y are evaluated at the optimum. When U^* is constant, we have $dU^* = 0$, or $U_x\, dx^* + U_y\, dy^* = 0$. From (12.42'), we have $\frac{U_x}{U_y} = \frac{P_x}{P_y}$ at the optimum. Thus we can also express $dU^* = 0$ by $P_x\, dx^* + P_y\, dy^* = 0$, or -$P_x\, dx^* - P_y\, dy^* = 0$.

7.

   (a) No; diminishing marginal utility means only that U_{xx} and U_{yy} are negative, but says nothing about U_{xy}. Therefore we cannot be sure that $\left|\bar{H}\right| > 0$ in (12.32) and $\frac{d^2y}{dx^2} > 0$ in (12.33').

   (b) No; if $\frac{dy^2}{dx^2} > 0$, and hence $\left|\bar{H}\right| > 0$, nothing definite be said about the sign of U_{xx} and U_{yy}, because U_{xy} also appears in $\left|\bar{H}\right|$.

Exercise 12.6

1. (a) $\sqrt{(jx)(jy)} = j = \sqrt{xy}$; homogeneous of degree one.

   (b) $\left[(jx)^2 - (jy)^2\right]^{\frac{1}{2}} = j\left(x^2 - y^2\right)^{\frac{1}{2}}$; homogeneous of degree one.

   (c) Not homogeneous.

   (d) $2jx + jy + 3\sqrt{(jx)(jy)} = j\left(2x + y + 3\sqrt{xy}\right)$; homogeneous of degree one.

   (e) $\frac{(jx)(jy)^2}{jw} + 2(jx)(jw) = j^2\left(\frac{xy^2}{w} + 2xw\right)$; homogeneous of degree two.

   (f) $(jx)^4 - 5(jy)(jw)^3 = j^4\left(x^4 - 5yw^3\right)$; homogeneous of degree four.

2. Let $j = \frac{1}{k}$, then $\frac{Q}{K} = f\left(\frac{K}{K}, \frac{L}{K}\right) = f\left(l, \frac{L}{K}\right) = \psi\left(\frac{L}{K}\right)$. Thus $Q = K\psi\left(\frac{L}{K}\right)$.

   (a) When $MPP_K = 0$, we have $L\frac{\partial Q}{\partial L} = Q$, or $\frac{\partial Q}{\partial L} = \frac{Q}{L}$, or $MPP_L = APP_L$.

   (b) When $MPP_L = 0$, we have $K\frac{\partial Q}{\partial K} = Q$, or $\frac{\partial Q}{\partial K} = \frac{Q}{K}$, or $MPP_K = APP_K$.

3. Yes, they are true:

4.

   (a) $APP_L = \phi(k)$; hence APP_L indeed can be plotted against k.

   (b) $MPP_K = \phi'(k) =$ slope of APP_L.

   (c) $APP_K = \frac{\phi(k)}{k} = \frac{APP_L}{k} = \frac{\text{ordinate of a point on the } APP_L \text{ curve}}{\text{abscissa of that point}}$ =slope of radius vector to the APP_L curve

(d) $MPP_L = \phi(k) - k\phi'(k) = APP_L - k \cdot MPP_K$

5.

b. $APP_L = Ak^\alpha$, thus the slope of $APP_L = A\alpha k^{\alpha-1} = MPP_K$.

c. Slope of a radius vector $= \frac{Ak^\alpha}{k} = Ak^{\alpha-1} = APP_K$.

d. $APP_L - k \cdot MPP_K = Ak^\alpha - kA\alpha k^{\alpha-1} = Ak^\alpha - A\alpha k^\alpha = A(1-\alpha)k^\alpha = MPP_L$.

6.

(a) Since the function is homogeneous of degree $(\alpha+\beta)$, if $\alpha+\beta > 1$, the value of the function will increase more than j-fold when K and L are increased j-fold, implying increasing returns to scale.

(b) If $\alpha+\beta < 1$, the value of the function will increase less than j-fold when K and L are increased j-fold, implying decreasing returns to scale.

(c) Taking the natural log of both sides of the function, we have $\ln Q = \ln A + \alpha \ln K + \beta \ln L$

Thus $\quad \epsilon_{QK} = \frac{\partial(\ln Q)}{\partial(\ln K)} = \alpha \quad$ and $\quad \epsilon_{QL} = \frac{\partial(\ln Q)}{\partial(\ln L)} = \beta$

7.

(a) $A(jK)^\alpha (jL)^b (jN)^c = j^{a+b+c} \cdot AK^\alpha L^b N^c = j^{a+b+c}Q$; homogeneous of degree $a+b+c$.

(b) $a+b+c = 1$ implies constant returns to scale; $a+b+c > 1$ implies increasing returns to scale.

(c) Share for $N = \frac{N\left(\frac{\partial Q}{\partial N}\right)}{Q} = \frac{NAK^aL^bcN^{c-1}}{AK^aL^bN^c} = c$

8.

(a) $j^2Q = g(jK, jL)$

(b) Let $j = \frac{1}{L}$. Then the equation in (a) yields:

$\frac{Q}{L^2} = g\left(\frac{K}{L}, 1\right) = \phi\left(\frac{K}{L}\right) = \phi(k)$. This implies that $Q = L^2\phi(k)$.

(c) $MPP_K = \frac{\partial Q}{\partial k} = L^2\phi'(k)\left(\frac{\partial k}{\partial K}\right) = L^2\phi'(k)\left(\frac{1}{L}\right) = L\phi'(k)$. Now MPP_K depends on L as well as k.

(d) If K and L are both increased j-fold in the MPP_K expression in (c), we get

$$(jL)\phi'\left(\frac{jK}{jL}\right) = jL\phi'\left(\frac{K}{L}\right) = jL\phi'(k) = j \cdot MPP_K$$

Thus MPP_K is homogeneous of degree one in K and L.

Exercise 12.7

1.

   (a) Linear homogeneity implies that the output levels of the isoquants are in the ratio of $1 : 2 : 3$ (from southwest to northeast).

   (b) With second-degree homogeneity, the output levels are in the ratio of $1 : 2^2 : 3^2$, or $1 : 4 : 9$.

2. Since $\left(\frac{b^*}{a^*}\right) = \left(\frac{\beta}{\alpha}\right)\left(\frac{P_a}{P_b}\right)$, it will plot as a straight line passing through the origin, with a (positive) slope equal to $\frac{\beta}{\alpha}$. This result does not depend on the assumption $\alpha + \beta = 1$. The elasticity of substitution is merely the elasticity of this line, which can be read (by the method of Fig. 8.2) to be unity at all points.

3. Yes, because Q_{LL} and Q_{KK} have both been found to be negative.

4. On the basis of (12.66), we have

$$\frac{d^2K}{dL^2} = \frac{d}{dL}\left[\frac{\delta-1}{\delta}\left(\frac{K}{L}\right)^{1+\rho}\right] = \frac{\delta-1}{\delta}(1+\rho)\left(\frac{K}{L}\right)^{\rho}\frac{d}{dL}\left(\frac{K}{L}\right)$$
$$= \frac{\delta-1}{\delta}(1+\rho)\left(\frac{K}{L}\right)^{\rho}\frac{1}{L^2}\left(L\frac{dK}{dL} - K\right) > 0 \qquad \left[\text{because } \frac{dK}{dL} < 0\right]$$

5.

   (a) $\frac{\text{Labor share}}{\text{Capital share}} = \frac{Lf_L}{Kf_k} = \frac{1-\delta}{\delta}\left(\frac{K}{L}\right)^{\rho}$. A larger ρ implies a larger capital share in relation to the labor share.

   (b) No; no.

6.

   (a) If $\rho = -1$, (12.66) yields $\frac{dK}{dL} = -\frac{(1-\delta)}{\delta} = constant$. The isoquants would be downward-sloping straight lines.

   (b) By (12.68), σ is not defined for $\rho = -1$. But as $\rho \to -1$, $\sigma \to \infty$.

   (c) Linear isoquants and infinite elasticity of substitution both imply that the two inputs are perfect substitutes.

7. If both K and L are changed j-fold, output will change from Q to:

   $A\left[\delta\,(jk)^{-\rho}+(1-\delta)\,(jL)^{-\rho}\right]^{-\frac{r}{\rho}}=A\left\{j^{-\rho}\left[\delta K^{-\rho}+(1-\delta)\,L^{-\rho}\right]\right\}^{-\frac{r}{\rho}}=\left(j^{-\rho}\right)^{-\frac{r}{\rho}}Q=j^{r}Q$

   Hence r denotes the degree of homogeneity. With $r>1$ ($r<1$), we have increasing (decreasing) returns to scale.

8. By L'Hôpital's Rule, we have:

   (a) $\lim_{x\to 4}\frac{x^2-x-12}{x-4}=\lim_{x\to 4}\frac{2x-1}{1}=7$

   (b) $\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{e^x}{1}=1$

   (c) $\lim_{x\to 0}\frac{5^x-e^x}{x}=\lim_{x\to 0}\frac{5^x\ln 5-e^x}{1}=\ln 5-1$

   (d) $\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0$

9.

   (a) By successive applications of the rule, we find that

   $\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}=\lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x}=...=\lim_{x\to\infty}\frac{n!}{e^x}=0$

   (b) By taking $m\,(x)=\ln x$, and $n\,(x)=\frac{1}{x}$, we have

   $\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0^+}-x=0$

   (c) Since $x^x=\exp\left(\ln x^x\right)=\exp\left(x\ln x\right)$, and since, from (b) above, the expression $x\ln x$ tends to zero as $x\to 0^+$, x^x must tend to $e^0=1$ as $x\to 0^+$.

CHAPTER 13

Exercise 13.1

1. For the minimization problem, the necessary conditions become

$$f'(x_1) = 0 \quad \text{and} \quad x_1 > 0$$
$$f'(x_1) = 0 \quad \text{and} \quad x_1 = 0$$
$$f'(x_1) > 0 \quad \text{and} \quad x_1 = 0$$

These can be condensed into the single statement

$$f'(x_1) > 0 \quad x_1 \geq 0 \quad \text{and} \quad x_1 f'(x_1) = 0$$

2.

(a) Since λ_i and $\partial Z/\partial\lambda_i$ are both nonnegative, each of the m component terms in the summation expression must be nonnegative, and there is no possibility for any term to be cancelled out by another, the way (-3) cancels out $(+3)$. Consequently, the summation expression can be zero if and only if every component term is zero. This is why the one-equation condition is equivalent to the m separate conditions taken together as a set.

(b) We can do the same for the conditions $x_j \dfrac{\partial Z}{\partial x_j} = 0$. This is because, for each j, $x_j \dfrac{\partial Z}{\partial x_j}$ must be nonpositive, so that no "cancellation" is possible.

3. The condition $x_j \dfrac{\partial Z}{\partial x_j} = 0$ $(j = 1, 2, \cdots, m)$ can be condensed, and so can the conditions $\lambda_i \dfrac{\partial Z}{\partial \lambda_i} = 0$ $(i = 1, 2, \cdots, m)$.

4. The expanded version of (13.19) is:

$$\frac{\partial Z}{\partial x_j} = f_j - \sum_{i=1}^{m} \lambda_i g_j^i \geq 0 \quad x_j \geq 0 \quad \text{and} \quad x_j(f_j - \sum_{i=1}^{m} \lambda_i g_j^i) = 0$$
$$\frac{\partial Z}{\partial \lambda_i} = r_i - g^i(x_1, \cdots, x_n) \leq 0, \quad \lambda_i \geq 0 \quad \text{and} \quad \lambda_i[r_i - g^i(x_1, \cdots, x_n)] = 0$$
$$(i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, n)$$

5.

$$\begin{array}{ll} \text{Maximize} & -C = -f(x_1, \cdots, x_n) \\ \text{Subject to} & -g^1(x_1, \cdots, x_n) \le -r_1 \\ & \quad \vdots \\ & -g^m(x_1, \cdots, x_n) \le -r_m \\ \text{and} & x_1, \cdots, x_n \ge 0 \end{array}$$

with the Lagrangian function in the form of

$$Z = -f(x_1, \cdots, x_n) + \sum_{i=1}^{m} \lambda_i[-r_i + g^i(x_1, \cdots, x_n)]$$

the Kuhn-Tucker conditions (13.16) yield

$$\frac{\partial Z}{\partial x_j} = -f_j + \sum_{i=1}^{m} \lambda_i g^i_j \le 0 \quad x_j \ge 0 \quad \text{and} \quad x_j \frac{\partial Z}{\partial x_j} = 0$$

$$\frac{\partial Z}{\partial \lambda_i} = -r_i + g^i(x_1, \cdots, x_n) \ge 0, \quad \lambda_i \ge 0 \quad \text{and} \quad \lambda_i \frac{\partial Z}{\partial \lambda_i} = 0$$

$$(i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, n)$$

These are identical with the results in the preceding problem.

Exercise 13.2

1. Since x_1^* and x_2^* are both nonzero, we may disregard (13.22), but (13.23) requires that:

$$6x_1(10 - x_1^2 - x_2)^2 dx_1 + 3(10 - x_1^2 - x_2)^2 dx_2 \le 0 \quad \text{and} \quad -dx_1 \le 0$$

The first inequality is automatically satisfied at the solution, and the second means that $dx_1 \ge 0$, with dx_2 free. Thus we may admit as a test vector, say $(dx_1, dx_2) = (1, 0)$, which plots as an arrow pointing eastward from the solution point in Fig. 13.3. No qualifying arc can be found for this vector.

2. The constraint border is a circle with a radius of 1, and with its center at $(0, 0)$. The optimal solution is at $(1, 0)$. By (13.22), the test vectors must satisfy $dx_2 \ge 0$. By (13.23), we must have $2x_1^* dx_1 + 2x_2^* dx_2 = 2dx_1 \le 0$. Thus the test vectors can only point towards due north, northwest, or due west. There does exist a qualifying arc for each such vector. (E.g., the constraint border itself can serve as a qualifying arc for the due-north test vector, as illustrated

in the accompanying diagram.)

The Lagrangian function and the Kuhn-Tucker conditions are:

$Z = x_1 + \lambda_1(1 - x_1^2 - x_2^2)$

$\partial Z/\partial x_1 = 1 - 2\lambda_1 x_1 \leq 0$ plus the nonnegativity and

$\partial Z/\partial x_2 = 1 - 2\lambda_1 x_2 \leq 0$ complementary slackness conditions

$\partial Z/\partial \lambda_1 = 1 - x_1^2 - x_2^2 \geq 0$

Since $x_1^* = 1$, $\partial Z/\partial x_1$ should vanish; thus $\lambda_1^* = 1/2$. This value of λ_1^*, together with the x_1^* and x_2^* values, satisfy all the Kuhn-Tucker conditions.

3. The feasible region consists of the points in the first quadrant lying on or below the curve $x_2 = x_1^2$. The optimal solution is at the point of origin, a cusp.

   Since $x_1^* = x_2^* = 0$, the test vectors must satisfy $dx_1 \geq 0$ and $dx_2 \geq 0$, by (13.20). Moreover, (13.23) shows that we must have $2x_1^* dx_1 - dx_2 = -dx_2 \geq 0$, or $dx_2 \leq 0$. The double requirement of $dx_2 \geq 0$ and $dx_2 \leq 0$ means that $dx_2 = 0$. Thus the test vectors must be horizontal, and pointing eastward (except for the null vector which does not point anywhere). Qualifying arcs clearly do exist for each such vector.

The Lagrangian function and the Kuhn-Tucker conditions are

$Z = x_1 + \lambda_1(-x_1^2 + x_2)$

$\partial Z/\partial x_1 = 1 - 2\lambda_1 x_1 \geq 0$ plus the nonnegativity and

$\partial Z/\partial x_2 = \lambda_1 \geq 0$ complementary slackness conditions

$\partial Z/\partial \lambda_1 = -x_1^2 + x_2 \leq 0$

At (0,0), the first and the third marginal conditions are duly satisfied. As long as we choose any value of $\lambda_1^* \geq 0$, all the Kuhn-Tucker conditions are satisfied despite the cusp.

4. (a) $Z = x_1 + \lambda_1[x_2 + (1 - x_1)^3]$

   Complementary slackness require that $\partial Z/\partial x_1$ vanish, but we actually find that, at the optimal solution (1,0), $\partial Z/\partial x_1 = 1 - 3\lambda_1(1 - x_1)^2 = 1$.

   (b) $Z_0 = \lambda_0 x_1 + \lambda_1[x_2 + (1 - x_1)^3]$

   The Kuhn-Tucker conditions are

   $\partial Z_0/\partial x_1 = \lambda_0 - 3\lambda_1(1 - x_1)^2 \geq 0$ plus the nonnegativity and

   $\partial Z_0/\partial x_2 = \lambda_1 \geq 0$ complementary slackness conditions

   $\partial Z_0/\partial \lambda_1 = x_2 + (1 - x_1)^3 \leq 0$

   By choosing $\lambda_0^* = 0$ and $\lambda_1^* \geq 0$, we can satisfy all of these conditions at the optimal solution.

Exercise 13.4

1.

(a) Maximize $-C = -F(x)$
subject to $-G^i(x) \leq r_i \quad (i = 1, 2, \cdots, m)$
and $x \geq 0$

(b) $f(x) = -F(x)$, and $g^i(x) = -G^i(x)$.

(c) $F(x)$ should be convex, and $G^i(x)$ should be concave, in the nonnegative orthant.

(d) Given the minimization program: Minimize $C = F(x)$, subject to $G^i(x) \geq r_i$, and $x \geq 0$, if (a) F is differentiable and convex in the nonnegative orthant. (b) each G^i is differentiable and concave in the nonnegative orthant, and (c) the point x^* satisfies the Kuhn-Tucker minimum conditions (13.17), then x^* gives a global minimum of C.

2. No. A unique saddle value should satisfy the strict inequality $Z(x, \lambda^*) < Z(x^*, \lambda^*) < Z(x^*, \lambda)$ because uniqueness precludes the possibility of $Z(x, \lambda^*) = Z(x^*, \lambda^*)$ and $Z(x^*, \lambda^*) = Z(x^*, \lambda)$.

(a) Applicable: $f(x)$ is linear and, hence, concave; and $g^1(x)$is convex because it is a sum of convex functions.

(b) Applicable: $f(x)$ is convex; and $g^1(x)$ is linear and, hence, concave.

(c) Inapplicable: $f(x)$ is linear and may thus be considered as convex, but $g^1(x)$ is convex too, which violates condition (b) for the minimization problem.

CHAPTER 14

Exercise 14.2

1.

(a) $\int 16x^{-3}dx = 16\frac{x^{-2}}{-2} + c = -8x^{-2} + c \qquad (x \neq 0)$

(b) $\int 9x^8 dx = x^9 + c$

(c) $\int x^5 dx = 3\int xdx = \frac{1}{6}x^6 - \frac{3}{2}x^2 + c$

(d) If $f(x) = -2x$, then $f'(x) = -2$. Using Rule IIa, we have
$-\int(-2)e^{-2x}dx = -\int f'(x)e^{f(x)}dx = -e^{f(x)} + c = -e^{-2x} + c$

(e) If $f(x) = x^2 + 1$, then $f'(x) = 2x$. Using Rule IIIa, we have
$2\int \frac{2x}{x^2+1}dx = 2\int \frac{f'(x)}{f(x)}dx = 2ln(f(x)) + c = 2ln(x^2+1) + c$

(f) Let $u = ax^2 + bx$. Then $\frac{du}{dx} = 2ax + b$. Thus
$\int \frac{du}{dx}u^7 dx = \int u^7 du = \frac{1}{8}u^8 + c = \frac{1}{8}(ax^2 + bx)^8 + c$

2.

(a) $13\int e^x dx = 13e^x + c$

(b) $3\int e^x dx + 4\int \frac{1}{x}dx = 3e^x + 4lnx + c \qquad (x > 0)$

(c) $5\int e^x dx + 3\int x^{-2}dx = 5e^x - 3x^{-1} + c \qquad (x \neq 0)$

(d) Let $u = -(2x+7)$. Then $\frac{du}{dx} = -2$. Thus
$3e^{-(2x+7)}dx = \int 3\left(-\frac{1}{2}\frac{du}{dx}\right)e^u dx = -\frac{3}{2}\int e^u du = -\frac{3}{2}e^u + c = -\frac{3}{2}e^{-(2x+7)} + c$

(e) Let $u = x^2 + 3$. Then $\frac{du}{dx} = 2x$, and
$\int 4xe^{x^2+3}dx = \int 2\frac{du}{dx}e^u dx = 2\int e^u du = 2e^u + c = 2e^{x^2+3} + c$

(f) Let $u = x^2 + 9$. Then $\frac{du}{dx} = 2x$, and
$\int xe^{x^2+9}dx = \int \frac{1}{2}\frac{du}{dx}e^u dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + c = \frac{1}{2}e^{x^2+9} + c$

3.

(a) $\int \frac{3dx}{x} = 3\int \frac{dx}{x} = 3ln|x| + c \qquad (x \neq 0)$

(b) Let $u = x - 2$. Then $\frac{du}{dx} = 1$, and $\int \frac{dx}{x-2} = \int \frac{1}{u}\frac{du}{dx}dx = \int \frac{1}{u}du = ln|u| + c = ln|x-2| + c \qquad (x \neq 2)$

(c) (c) Let $u = x^2 + 3$. Then $\frac{du}{dx} = 2x$, and
$\int \frac{du}{dx}\frac{1}{u}dx = \int \frac{du}{u} = lnu + c = ln(x^2+3) + c$

(d) Let $u = 3x^2 + 5$. Then $\frac{du}{dx} = 6x$, and
$\int \left(\frac{1}{6}\frac{du}{dx}\right)\frac{1}{u}dx = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}lnu + c = \frac{1}{6}ln(3x^2+5) + c$

4.

(a) Let $v = x + 3$, and $u = \frac{2}{3}(x+1)^{\frac{3}{2}}$, so that $dv = dx$ and $du = (x+1)^{\frac{1}{2}}dx$. Then, by Rule VII, we have
$\int (x+3)(x+1)^{\frac{1}{2}}dx = \frac{2}{3}(x+1)^{\frac{3}{2}}(x+3) - \int \frac{2}{3}(x+1)^{\frac{2}{3}}dx = \frac{2}{3}(x+1)^{\frac{2}{3}}(x+3) - \frac{4}{15}(x+1)^{\frac{5}{2}} + c$

(b) Let $v = lnx$ and $u = \frac{1}{2}x^2$, so that $dv = \frac{1}{x}dx$ and $du = xdx$.
Then Rule VII gives us
$\int xlnxdx = \frac{1}{2}x^2lnx - \int \frac{1}{2}x^2\frac{1}{x}dx = \frac{1}{2}x^2lnx - \frac{1}{4}x^2 + c = \frac{1}{4}x^2(2lnx - 1) + c \qquad (x > 0)$

5. $\int [k_l f_l(x) + ... + k_n f_n]dx = \int k_l f_l dx + ... + \int k_n f_n dx = k_l \int f_l(x)dx + ... + k_n \int f_n(x)dx = \sum_{i=1}^{n} k_i \int f_i(x)dx$

Exercise 14.3

1.

(a) $\frac{1}{6}\ x^3\Big]_1^3 = \frac{1}{6}\left(3^3 - 1^3\right) = \frac{26}{6} = 4\frac{1}{3}$

(b) $\left[\frac{x^4}{4} + 3x^2\right]_0^1 = \left(\frac{1}{4} + 3\right) - 0 = 3\frac{1}{4}$

(c) $2x^{\frac{3}{2}}\Big]_1^3 = 2\sqrt{x^3}\Big]_1^3 = 2\sqrt{27} - 2$

(d) $\left[\frac{x^4}{4} - 2x^3\right]_2^4 = \left[\frac{4^4}{4} - 2(4)^3\right] - \left[\frac{2^4}{4} - 2(2)^3\right] = (64 - 128) - (4 - 16) = -52$

(e) $\left[\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right]_{-1}^1 = \left(\frac{a}{3} + \frac{b}{2} + c\right) - \left(-\frac{a}{3} + \frac{b}{2} - c\right) = 2\left(\frac{a}{3} + c\right)$

(f) $\frac{1}{2}\left(\frac{x^3}{3} + 1\right)^2\Bigg]_4^2 = \frac{1}{2}\left[\left(\frac{8}{3} + 1\right)^2 - \left(\frac{64}{3} + 1\right)^2\right] = \frac{1}{2}\frac{121-4489}{9} = -\frac{4368}{18} = -242\frac{2}{3}$

2.

(a) $-\frac{1}{2}e^{-2x}\big]_1^2 = -\frac{1}{2}\left(e^{-4} - e^{-2}\right) = \frac{1}{2}\left(e^{-2} - e^{-4}\right)$

(b) $ln|x + 2|]_{-1}^{e-2} = lne - ln1 = 1 - 0 = 1$

(c) $\left[\frac{1}{2}e^{2x} + e^x\right]_2^3 = \left(\frac{1}{2}e^6 + e^3\right) - \left(\frac{1}{2}e^4 + e^2\right) = e^2\left(\frac{1}{2}e^4 - \frac{1}{2}e^2 + e - 1\right)$

(d) $[ln|x| + ln|1 + x|]_e^6 = [ln|x(1 + x)|]_e^6$

3.

(a) $A\star\star = \sum_{i=1}^4 f(x_{i+1})\triangle x_i$

(b) $A\star\star < A_i$ underestimate.

(c) $A\star\star$ would approach A.

(d) Yes.

(e) $f(x)$ is Riemann integrable.

4. The curve refers to the graph of the integrand $f(x)$. If we plot the graph of $F(x)$, the definite integral – which has the value $F(b) - F(a)$ – will show up there as a vertical distance.

5.

(a) $\frac{c}{b}x\big]_0^b = c - 0 = c$

(b) $t]_0^c = c - 0 = c$

Exercise 14.4

1. None is improper.

2. (a) and (d) each has an infinite limit of integration: (c) and (e) have infinite integrands, at x = 0 and x = 2, respectively.

3.

   (a) $\int_0^\infty e^{-rt}dt = \lim_{b\to\infty} \int_0^b e^{-rt}dt = \lim_{b\to\infty} \left[-\frac{1}{r}e^{-rt}\right]_0^b = \lim_{b\to\infty} -\frac{1}{r}\left(e^{-rb} - e^0\right) = -\frac{1}{r}(0-1) = \frac{1}{r}$

   c. $\int_0^1 x^{-\frac{2}{3}}dx = \lim_{a\to 0^+} \int_a^1 x^{-\frac{2}{3}}dx = \lim_{a\to 0^+} \left[3x^{\frac{1}{3}}\right]_a^1 = 3$

   d. $\int_{-\infty}^0 e^{rt}dt = \lim_{a\to -\infty} \left[\frac{1}{r}e^{rt}\right]_a^0 = \frac{1}{r}(1-0) = \frac{1}{r}$

   e. $\int_1^5 \frac{dx}{x-2} = \int_1^2 \frac{dx}{x-2} + \int_2^5 \frac{dx}{x-2}$

   $= \lim_{b\to 2^-} [\ln|x-2|]_1^b + \lim_{a\to 2^+} [\ln(x-2)]_a^5 = I_1 + I_2$

   $I_1 = \lim_{b\to 2^-} [\ln|b-2| - \ln 1] = -\infty$; and $I_2 = \lim_{a\to 2^+} [\ln 3 - \ln(a-2)] = +\infty$

   This integral is divergent. (I_1 and I_2 cannot cancel each other out.)

4. $I_2 = \lim_{a\to 0^+} \int_a^1 x^{-3}dx = \lim_{a\to 0^+} \left[-\frac{1}{2}x^{-2}\right]_a^1 = \lim_{a\to 0^+} \left(-\frac{1}{2} + \frac{1}{2a^2}\right) = +\infty$

5.

   (a)

   (b) $Area = \int_0^\infty ce^{-t}dt = \lim_{b\to +\infty} [-ce^{-t}]_0^b = c \quad (finite)$

Exercise 14.5

1.

(a) $R(Q) = \int \left(28Q - e^{0.3Q}\right) dQ = 14Q^2 - \frac{10}{3}e^{0.3Q} + c$. The initial condition is R(0) =0. Setting $Q = 0$ in $R(Q)$, we find $R(0) = -\frac{10}{3} + c$. Thus $c = \frac{10}{3}$. And $R(Q) = 14Q^2 - \frac{10}{3}e^{0.3Q} + \frac{10}{3}$.

(b) $R(Q) = \int 10(1+Q)^{-2}dQ$. Let $u = 1 + Q$. Then $\frac{du}{dQ} = 1$, or $du = dQ$, and $R(Q) = 10\int u^{-2}du = 10\left(-u^{-1}\right) + c = -10(1+Q)^{-1} + c$. Since $R(0) = 0$, we have $0 = -10 + c$, or $c = 10$. Thus

$$R(Q) = \frac{-10}{1+Q} + 10 = \frac{= -10 + 10 + 10Q}{1+Q} = \frac{100}{1+Q}$$

2.

(a) $M(Y) = \int 0.1dY = 0.1Y + c$. From the initial condition, we have $20 = 0.1(0) + c$, giving us $c = 20$. Thus $M(Y) = 0.1Y + 20$.

(b) $C(Y) = \int \left(0.8 + 0.1Y^{-\frac{1}{2}}\right) dY = 0.8Y + 0.2Y^{\frac{1}{2}} + c$. From the side information, we have $100 = 0.8(100) + 0.2(100)^{\frac{1}{2}} + c$, or $c = 18$. Thus $C(Y) = 0.8Y + 0.2Y^{\frac{1}{2}} + 18$.

3.

(a) $K(t) = \int 12t^{\frac{1}{3}}dt = 9t^{\frac{4}{3}} + c$. Since $K(0) = 25$, we have $25 = 9(0) + c$, so that $c = 25$. Thus $K(t) = 9t^{\frac{4}{3}} + 25$.

(b) $K(1) - K(0) = 9t^{\frac{4}{3}}\Big]_0^1 = 9$; $K(3) - K(1) = 9\left[t^{\frac{4}{3}}\right]_1^3 = 9\left[3\left(3^{\frac{1}{3}}\right) - 1\right]$

4.

(a) $\Pi = \frac{1000}{0.05}\left[1 - e^{-0.05(2)}\right] = 20,000\left[1 - e^{-0.10}\right] = 20,000(0.0952) = 1904.00$ (*approximately*)

(b) $\Pi = \frac{1000}{0.04}\left[1 - e^{-0.04(3)}\right] = 25,000\left[1 - e^{-0.12}\right] = 25,000(0.1131) = 2827.50$ (*approximately*)

5.

(a) $\Pi = \$\frac{1450}{0.05} = \$29,000$

(b) $\Pi = \$\frac{2460}{0.08} = \$30,750$

Exercise 14.6

1. Capital alone is considered. More specifically, the production function is $\kappa = \rho K$. The constancy of the capacity-capital ratio ρ means that the output level is a specific multiple of the amount of capital used. Since the production process obviously requires the labor factor as well, the equation above implies that labor and capital are combined in a fixed proportion, for only then can we consider capital alone to the exclusion of labor. This also seems to carry the implication of a perfectly elastic supply of labor.

2. The second equation in (13.16) states that the rate of growth of I is constant ρs. Thus the investment function should be $I(t) = Ae^{\rho st}$, where A can be definitized to $I(0)$.

3. If $I < 0$, then $|I| = -I$. Thus the equation $|I| = Ae^{\rho st}$ becomes $-I = Ae^{\rho st}$. Setting $t = 0$, we find that $-I(0) = Ae^0 = A$. Thus we now have $-I = -I(0)e^{\rho st}$, which is identical with (13.18).

4. Left side $= \int_0^t \frac{1}{I}\frac{dI}{dt}dt = \int_{I(0)}^{I(t)} \frac{1}{I}dI = \ln I]_{I(0)}^{I(t)}$

   $\ln I(t) = \ln I(0) = \ln \frac{I(t)}{I(0)}$

   Right side$= \int_0^t \rho s\ dt = \rho st]_0^t = \rho st$

   Equating the two sides, we have $\ln \frac{I(t)}{I(0)} = \rho st$. Taking the antilog (letting each side become the exponent of e), we have

   $\exp\left[\ln \frac{I(t)}{I(0)}\right] = \exp(\rho st)$

   or $\frac{I(t)}{I(0)} = e^{\rho st}$

   or $I(t) = I(0)e^{\rho st}$

CHAPTER 15

Exercise 15.1

1.

(a) With $a = 4$ and $b = 12$, we have $y_c = Ae^{-4t}$, $y_p = \frac{12}{4} = 3$. The general solution is $y(t) = Ae^{-4t} + 3$. Setting $t = 0$, we get $y(0) = A + 3$, thus $A = Y(0) - 3 = 2 - 3 = -1$. The definite solution is $y(t) = -e^{-4t} + 3$.

(b) $y_c = Ae^{-(-2)t}$, $y_p = \frac{0}{-2} = 0$. The general solution is $y(t) = Ae^{2t}$. Setting $t = 0$, we have $y(0) = A$; i.e., $A = 9$. Thus the definite solution is $y(t) = 9e^{2t}$.

(c) $y_c = Ae^{-10t}$, $y_p = \frac{15}{10} = \frac{3}{2}$. Thus $y(t) = Ae^{-10t} + \frac{3}{2}$. Setting $t = 0$, we get $y(0) = A + \frac{3}{2}$, i.e., $A = y(0) - \frac{3}{2} = 0 - \frac{3}{2} = -\frac{3}{2}$.
The definite solution is $y(t) = \frac{3}{2}\left(1 - e^{-10t}\right)$.

(d) Upon dividing through by 2, we get the equation $\frac{dy}{dt} + 2y = 3$. Hence $y_c = Ae^{-2t}$, $y_p = \frac{3}{2}$, and $y(t) = Ae^{-2t} + \frac{3}{2}$. Setting $t = 0$, we get $y(0) = A + \frac{3}{2}$, implying that $A = y(0) - \frac{3}{2} = 0$. The definite solution is $y(t) = \frac{3}{2}$.

3.

(a) $y(t) = (0 - 4)e^{-t} + 4 = 4\left(1 - e^{-t}\right)$

(b) $y(t) = 1 + 23t$

(c) $y(t) = (6 - 0)\, e^{5t} + 0 = 6e^{5t}$

(d) $y(t) = \left(4 - \frac{2}{3}\right) e^{-3t} + \frac{2}{3} = 3\frac{1}{3}e^{-3t} + \frac{2}{3}$

(e) $y(t) = [7 - (-1)]\, e^{7t} + (-1) = 8e^{7t} - 1$

(f) After dividing by 3 throughout, we find the solution to be

$$y(t) = \left(0 - \frac{5}{6}\right) e^{-2t} + \frac{5}{6} = \frac{5}{6}\left(1 - e^{-2t}\right)$$

Exercise 15.2

1. The D curve should be steeper then the S curve. This means that $|-\beta| > |\delta|$, or $-\beta < \delta$, which is precisely the criterion for dynamic stability.

2. From (14.9), we may write $\alpha + \gamma = (\beta + \delta)\bar{P}$. Hence (14.10') can be rewritten as $\frac{dP}{dt} + j(\beta + \delta)P = j(\beta + \delta)\bar{P}^*$, or $\frac{dP}{dt} + kP = k\bar{P}$, or $\frac{dP}{dt} + k\left(P - \bar{P}\right) = 0$. By (14.3'), the time path corresponding to this homogeneous differential equation is $\Delta(t) = \Delta(0)e^{-kt}$. If $\Delta(0) \neq 0$, then $\Delta(t) \equiv P(t) - \bar{P}$ will converge to zero if and only if $k > 0$. This conclusion is no different from the one stated in the text.

3. The price adjustment equation (14.10) is what introduces a derivative (pattern of change) into the model, thereby generating a differential equation.

4.

   (a) By substitution, we have $\frac{dP}{dt} = j(Q_d - Q_s) = j(\alpha + \gamma) - j(\beta + \delta)P + j\sigma\frac{dP}{dt}$. This can be simplified to

   $$\frac{dP}{dt} + \frac{j(\beta+\delta)}{1-j\sigma}P = \frac{j(\alpha+\gamma)}{1-j\sigma} \qquad (1 - j\sigma \neq 0)$$

   The general solution is, by (14.5),

   $$P(t) = A\exp\left[-\frac{j(\beta+\delta)}{1-j\sigma}t\right] + \frac{\alpha+\gamma}{\beta+\delta}$$

   (b) Since $\frac{dP}{dt} = 0$ iff $Q_d = Q_s$, then the inter-temporal equilibrium price is the same as the market-clearing equilibrium price $\left(= \frac{\alpha+\gamma}{\beta+\delta}\right)$.

   (c) Condition for dynamic stability: $1 - j\sigma > 0$, or $\sigma < \frac{1}{j}$.

5.

   (a) Setting $Q_d = Q_s$, and simplifying, we have

   $$\frac{dP}{dt} + \frac{\beta+\delta}{\eta}P = \frac{\alpha}{\eta}$$

   The general solution is, by (14.5),

   $$P(t) = A\exp\left(-\frac{\beta+\delta}{\eta}t\right) + \frac{\alpha}{\beta+\delta}$$

   (b) Since -$\frac{\beta+\delta}{\eta}$ is negative, the exponential term tends to zero, as t tends to infinity. The inter-temporal equilibrium is dynamically stable.

(c) Although there lacks a dynamic adjustment mechanism for price, the demand function contains a $\frac{dP}{dt}$ term. This gives rise to a differential equation and makes the model dynamic.

Exercise 15.3

We shall omit all constants of integration in this Exercise.

1. Since $u = 5$, $w = 15$, and $\int u\,dt = 5t$, solution formula (14.15) gives

   $$y(t) = e^{-5t}\left(A + \int 15e^{5t}dt\right) = e^{-5t}\left(A + 3e^{5t}\right) = Ae^{-5t} + 3$$

   The same result can be obtained also by using formula (14.5).

2. Since $u = 2t$, $w = 0$, and $\int u\,dt = t^2$, solution formula (14.14) gives us $y(t) = Ae^{-t^2}$.

3. Since $u = 2t$, $w = t$, and $\int u\,dt = t^2$, formula (14.15) yields

   $$y(t) = e^{-t^2}\left(A + \int te^{t^2}dt\right) = e^{-t^2}\left(A + \tfrac{1}{2}e^{t^2}\right) = Ae^{-t^2} + \tfrac{1}{2}$$

   Setting $t = 0$, we find $y(0) = A + \frac{1}{2}$; i.e., $A = y(0) - \frac{1}{2} = 1$. Thus the definite solution is $y(t) = e^{-t^2} + \frac{1}{2}$.

4. Since $u = t^2$, $w = 5t^2$, and $\int u\,dt = \frac{t^3}{3}$, formula (14.15) gives us

   $$y(t) = e^{-\frac{t^3}{3}}\left(A + \int 5t^2e^{\frac{t^3}{3}}dt\right) = e^{-\frac{t^3}{3}}\left(A + 5e^{\frac{t^3}{3}}\right) = Ae^{-\frac{t^3}{3}} + 5$$

   Setting $t = 0$, we find $y(0) = A + 5$; thus $A = y(0) - 5 = 1$. The definite solution is $y(t) = e^{-\frac{t^3}{3}} + 5$.

5. Dividing through by 2, we get $\frac{dy}{dt} + 6y = -e^t$. Now with $u = 6$, $w = -e^t$, and $\int u\,dt = 6t$, formula (14.15) gives us

   $$y(t) = e^{-6t}\left(A + \int -e^te^{6t}dt\right) = e^{-6t}\left(A - \tfrac{1}{7}e^{7t}\right) = Ae^{-6t} - \tfrac{1}{7}e^t$$

   Setting $t = 0$, we find $y(0) = A - \frac{1}{7}$; i.e., $A = y(0) + \frac{1}{7} = 1$. The definite solution is $y(t) = e^{-6t} - \frac{1}{7}e^t$

6. Since $u = 1$, $w = t$, and $\int u\,dt = t$, the general solution is

   $$\begin{aligned} y(t) &= e^{-t}\left(A + \int te^tdt\right) \\ &= e^{-t}\left[A + e^t(t-1)\right] \qquad \text{[by Example 17, Section 13.2]} \\ &= Ae^{-t} + t - 1 \end{aligned}$$

Exercise 15.4

1.

(a) With $M = 2yt^3$ and $N = 3y^2t^2$, we have $\frac{\partial M}{\partial t} = 6yt^2 = \frac{\partial N}{\partial y}$.

Step i: $F(y,t) = \int 2yt^3 dy + \psi(t) = y^2t^3 + \psi(t)$

Step ii: $\frac{\partial F}{\partial t} = 3y^2t^2 + \psi'(t) = N = 3y^2t^2$; thus $\psi'(t) = 0$

Step iii: $\psi(t) = \int 0\, dt = k$

Step iv: $F(y,t) = y^2t^3 + k$, so the general solution is

$$y^2t^3 = c \qquad \text{or} \qquad y(t) = \left(\frac{c}{t^3}\right)^{\frac{1}{2}}$$

(b) With $M = 3y^2t$ and $N = y^3 + 2t$, we have $\frac{\partial M}{\partial t} = 3y^2 = \frac{\partial N}{\partial y}$.

Step i: $F(y,t) = \int 3y^2t\, dy + \psi(t) = y^3t + \psi(t)$

Step ii: $\frac{\partial F}{\partial t} = y^3 + \psi'(t) = N = y^3 + 2t$; thus $\psi'(t) = 2t$

Step iii: $\psi(t) = \int 2t\, dt = t^2$ [constant omitted]

Step iv: $F(y,t) = y^3t + t^2$, so the general solution is

$$y^3t + t^2 = c \quad \text{or} \qquad y(t) = \left(\frac{c - t^2}{t}\right)^{\frac{1}{3}}$$

(c) With $M = t(1 + 2y)$ and $N = y(1 + y)$, we have $\frac{\partial M}{\partial t} = 1 + 2y = \frac{\partial N}{\partial y}$.

Step i: $F(y,t) = \int t(1 + 2y)dy + \psi(t) = yt + y^2t + \psi(t)$

Step ii: $\frac{\partial F}{\partial t} = y + y^2 + \psi'(t) = N = y(1 + y)$; thus $\psi'(t) = 0$

Step iii: $\psi(t) = \int 0\, dt = k$

Step iv: $F(y,t) = yt + y^2t + k$, so the general solution is

$$yt + y^2t = c$$

(d) The equation can be rewritten as $4y^3t^2dy + \left(2y^4t + 3t^2\right) dt = 0$, with $M = 4y^3t^2$ and $N = 2y^4t + 3t^2$, so that $\frac{\partial M}{\partial t} = 8y^3t = \frac{\partial N}{\partial y}$.

Step i: $F(y,t) = \int 4y^3t^2\, dy + \psi(t) = y^4t^2 + \psi(t)$

Step ii: $\frac{\partial F}{\partial t} = 2y^4t + \psi'(t) = N = 2y^4t + 3t^2$; thus $\psi'(t) = 3t^2$

Step iii: $\psi(t) = \int 3t^2\, dt = t^3$ [constant omitted]

Step iv: $F(y,t) = y^4t^2 + t^3$, so the general solution is

$$y^4t^2 + t^3 = c \quad \text{or} \qquad y(t) = \left(\frac{c - t^3}{t^2}\right)^{\frac{1}{4}}$$

2.

(a) Inexact; y is an integrating factor.

(b) Inexact; t is an integrating factor.

3. Step i: $F(y,t) = \int M\,dy + \psi(t)$

Step ii: $\frac{\partial F}{\partial t} = \frac{\partial}{\partial t}\int M\,dy + \psi'(t) = N$; thus $\psi'(t) = N - \frac{\partial}{\partial t}\int M\,dy$

Step iii: $\psi(t) = \int \left(N - \frac{\partial}{\partial t}\int M dy\right) dt = \int N\,dt - \int \left(\frac{\partial}{\partial t}\int M\,dy\right) dt$

Step iv: $F(y,t) = \int M\,dy + \int N\,dt - \int \left(\frac{\partial}{\partial t}\int M\,dy\right) dt$

Setting $F(y,t) = c$, we obtain the desired result.

Exercise 15.5

1. (a)
   i. Separable; we can write the equation as $\frac{2}{y}dy + \frac{2}{t}dt = 0$.
   ii. Rewritten as $\frac{dy}{dt} + \frac{1}{t}y = 0$, the equation is linear.

(b)

   i. Separable; multiplying by $(y+t)$, we get $y\,dy + 2t\,dt = 0$.
   ii. Rewritten as $\frac{dy}{dt} = -2ty^{-1}$, the equation is a Bernoulli equation with $R = 0$, $T = -2t$ and $m = -1$. Define $z = y^{1-m} = y^2$. Then we can obtain from (14.24') a linearized equation

   $dz - 2(-2)t\,dt = 0$ or $\frac{dz}{dt} + 4t = 0$

(c)

   i. Separable; we can write the equation as $y\,dy + t\,dt = 0$.
   ii. Reducible; it is a Bernoulli equation with $R = 0$, $T = -t$, and $m = -1$.

(d)

   i. Separable; we can write the equation as $\frac{1}{3y^2}dy - t\,dt = 0$
   ii. Yes; it is a Bernoulli equation with $R = 0$, $T = 3t$, $m = 2$.

2.

(a) Integrating $\frac{2}{y}dy + \frac{2}{t}dt = 0$ after cancelling the common factor 2, we get $\ln y + \ln t = c$, or $\ln yt = c$. The solution is

$$yt = e^c = k \qquad \text{or} \qquad y(t) = \frac{k}{t}$$

Check: $\frac{dy}{dt} = -kt^{-2} = -\frac{y}{t}$ (consistent with the given equation).

(b) Cancelling the common factor $\frac{1}{y+t}$, and integrating we get $\frac{1}{2}y^2 + t^2 = c$. Thus the solution is

$$y(t) = \left(2c - 2t^2\right)^{\frac{1}{2}} = \left(k - 2t^2\right)^{\frac{1}{2}}$$

Check: $\frac{dy}{dt} = \frac{1}{2}\left(k - 2t^2\right)^{-\frac{1}{2}}(-4t) = -\frac{2t}{y}$ (which is equivalent to the given differential equation).

3. Integrating $y\,dy + t\,dt = 0$, we get $\frac{1}{2}y^2 + \frac{1}{2}t^2 = c$, or $y^2 + t^2 = 2c = A$. Thus the solution is $y(t) = \left(A - t^2\right)^{\frac{1}{2}}$. Treating it as a Bernoulli equation with $R = 0$, $T = -t$, $m = -1$, and $z = y^{1-m} = y^2$, we can use formula (14.24') to obtain the linearized equation $dz + 2t\,dt = 0$, or $\frac{dz}{dt} = -2t$, which has the solution $z = A - t^2$. Reverse substitution then yields the identical answer

$$y^2 = A - t^2 \qquad \text{or} \qquad y(t) = \left(A - t^2\right)^{\frac{1}{2}}$$

4. Integrating $\frac{1}{3}y^{-2}dy - t\,dt = 0$, we obtain $-\frac{1}{3}y^{-1} - \frac{1}{2}t^2 = c$, or $y^{-1} = -3c - \frac{3}{2}t^2 = A - \frac{3}{2}t^2$. The solution is $y(t) = \frac{1}{A - \frac{3}{2}t^2}$.

Treating it as a Bernoulli equation, on the other hand, we have $R = 0$, $T = 3t$, and $m = 2$. Thus we can write $dz + 3t\,dt = 0$, or $\frac{dz}{dt} = -3t$, which has the solution $z = A - \frac{3}{2}t^2$. Since $z = y^{1-m} = y^{-1}$, we have $y(t) = \frac{1}{z}$, which represents an identical solution.

5. The derivative of the solution is $\frac{dz}{dt} = 2At + 2$. The linearized equation itself implies on the other hand that $\frac{dz}{dt} = \frac{2z}{t} - 2$. But since $z = At^2 + 2t$, the latter result amounts to $2(At + 2) - 2 = 2At + 2$. Thus the two results are identical.

Exercise 15.6

1. (a) and (d): The phase line is upward-sloping, and the equilibrium is accordingly dynamically unstable.

(b) and (c): The phase line is downward-sloping, and the equilibrium is dynamically stable.

2.

(a) The phase line is upward-sloping for nonnegative y; the equilibrium $y^* = 3$ is dynamically unstable.

(b) The phase line slopes upward from the point of origin, reaches a peak at the point $\left(\frac{1}{4}, \frac{1}{16}\right)$, and then slopes downward thereafter. There are two equilibriums, $y^* = 0$ and $y^* = \frac{1}{2}$; the former is dynamically unstable, but the latter is dynamically stable.

3.

(a) An equilibrium can occur only when $\frac{dy}{dt} = 0$, i.e., only when $y = 3$, or $y = 5$.

(b) $\frac{d}{dy}\left(\frac{dy}{dt}\right) = 2y - 8 = \begin{cases} -2 & \text{when } y = 3 \\ +2 & \text{when } y = 5 \end{cases}$

Since this derivative measures the slope of the phase line, we can infer that the equilibrium at $y = 3$ is dynamically stable, but the equilibrium at $y = 5$ is dynamically unstable.

Exercise 15.7

1. Upon dividing by k throughout, the equation becomes

   $\frac{\dot{k}}{k} = \frac{s\,\phi(k)}{k} = \lambda$

   Since the first term on the right is equal to $\frac{s\frac{Q}{L}}{\frac{K}{L}} = \frac{sQ}{K} = \frac{\dot{K}}{K}$, the equation above means that:

   growth rate of $\frac{K}{L}$ = growth rate of $K-$ growth rate of L

2. $I \equiv \frac{dK}{dt} = \frac{d}{dt}Ae^{\lambda t} = Be^{\lambda t}$. Thus net investment, I, obviously also grows at the rate λ.

3. The rate of growth of Q should be the sum of the rates of growth of $T(t)$ and of $f(K, L)$. The former rate is given to be ρ; the latter rate is λ. Hence we have $r_Q = \rho + \lambda$.

4. The assumption of linear homogeneity (constant returns to scale) is what enables us to focus on the capital-labor ratio.

5.

(a) There is a single equilibrium $\bar{y}$ which lies between 1 and 3, and is dynamically stable.

(b)

There are two equilibriums: $\bar{y}_1$ (negative) is dynamically stable, and $\bar{y}_2$ (positive) is dynamically unstable

CHAPTER 16

Exercise 16.1

1. (a) By (16.3), $y_p = \frac{b}{a_2} = \frac{2}{5}$ (b) By $\left(16.3'\right)$, $y_p = \frac{bt}{a_1} = 7t$
   (c) By (16.3), $y_p = 9/3 = 3$ (d) By (16.3), $y_p = -4/-1 = 4$
   (e) By $\left(16.3''\right)$, $y_p = \frac{bt^2}{2} = 6t^2$

2.
   (a) With $a_1 = 3$ and $a_2 = -4$, we find $r_1, r_2 = \frac{1}{2}(-3 \pm 5) = 1, -4$. Thus $y_c = A_1e^t + A_2e^{-4t}$

   (b) With $a_1 = 6$ and $a_2 = 5$, we find $r_1, r_2 = \frac{1}{2}(-6 \pm 4) = -1, -5$. Thus $y_c = A_1e^{-t} + A_2e^{-5t}$

   (c) With $a_1 = -2$ and $a_2 = 1$, we get repeated roots $r_1 = r_2 = 1$. Thus, by (16.9) $y_c = A_3e^t + A_4te^t$

   (d) With $a_1 = 8$ and $a_2 = 16$, we find $r_1 = r_2 = -4$. Thus we have $y_c = A_3e^{-4t} + A_4te^{-4t}$

3.
   (a) By (16.3), $y_p = -3$. Adding this to the earlier-obtained y_c, we get the general solution $y(t) = A_1e^t + A_2e^{-4t} - 3$. Setting $t = 0$ in this solution, and using the first initial condition, we have $y(0) = A_1 + A_2 - 3 = 4$. Differentiating $y(t)$ and then setting $t = 0$, we find via the second initial condition that $y'(0) = A_1 - 4A_2 = 2$. Thus $A_1 = 6$ and $A_2 = 1$. The definite solution is $y(t) = 6e^t + e^{-4t} - 3$.

   (b) $y_p = 2$. The general solution is $y(t) = A_1e^{-t} + A_2e^{-5t} + 2$. The initial condition give us $y(0) = A_1 + A_2 + 2 = 4$, and $y'(0) = -A_1 - 5A_2 = 2$. Thus $A_1 = 3$ and $A_2 = -1$. The definite solution is $y(t) = 3e^{-t} - e^{-5t} + 2$.

   (c) $y_p = 3$. The general solution is $y(t) = A_3e^t + A_4te^t + 3$. Since $y(0) = A_3 + 3 = 4$, and $y'(0) = 1 + A_4 = 2$, we have $A_3 = 1$, and $A_4 = 1$. The definite solution is $y(t) = e^t + te^t + 3$.

   (d) $y_p = 0$. The general solution is $y(t) = A_3e^{-4t} + A_4te^{-4t}$. Since $y(0) = A_3 = 4$, and $y'(0) = -4A_3 + A_4 = 2$, we have $A_3 = 4$, and $A_4 = 18$. Thus, the definite solution is $y(t) = 4e^{-4t} + 18te^{-4t}$.

4. (a) Unstable (b) Stable (c) Unstable (d) Stable

(a) Setting $t = 0$ in the solution, we get $y(0) = 2 + 0 + 3 = 5$. This satisfies the first initial condition. The derivative of the solution is $y'(t) = -6e^{-3t} - 3te^{-3t} + e^{-3t}$, implying that $y'(0) = -6 - 0 + 1 = -5$. This checks with the second initial condition

(b) The second derivative is $y"(t) = 18e^{-3t} + 9te^{-3t} - 3e^{-3t} - 3e^{-3t} = 12e^{-3t} + 9te^{-3t}$. Substitution of the expressions for $y", y'$ and y into the left side of the differential equation yields the value of 27, since all exponential terms cancel out. Thus the solution is validated.

5. For the case of $r < 0$, we first rewrite te^{rt} as t/e^{-rt}, where both the numerator and the denominator tend to infinity as t tends to infinity. Thus, by L'Hôpital's rule,

$$\lim_{t\to\infty} \frac{t}{e^{-rt}} = \lim_{t\to\infty} \frac{1}{-re^{-rt}} = 0 \quad (\text{case of } r < 0)$$

For the case of $r > 0$, both component t and the component e^{rt} will tend to infinity. Thus their product te^{rt} will also tend to infinity.

For the case of $r = 0$, we have $te^{rt} = te^0 = t$. Thus te^{rt} tends to infinity as t tends to infinity

Exercise 16.2

1. (a) $r_1, r_2 = \frac{1}{2}(3 \pm \sqrt{-27}) = \frac{3}{2} \pm \frac{3}{2}\sqrt{3}i$
   (b) $r_1, r_2 = \frac{1}{2}(-2 \pm \sqrt{-64}) = -1 \pm 4i$
   (c) $x_1, x_2 = \frac{1}{4}(-1 \pm \sqrt{-63}) = -\frac{1}{4} \pm \frac{3}{4}\sqrt{7}i$
   (d) $x_1, x_2 = \frac{1}{4}(1 \pm \sqrt{-7}) = \frac{1}{4} \pm \frac{1}{4}\sqrt{7}i$

2. (a) Since 180 degree = 3.14159 radians,
   1 radian $= \frac{180}{3.14159}$ degrees = 57.3 degrees (or $57°18'$)
   (b) Similarly, 1 degree $= \frac{3.14159}{180}$ radians = 0.01745 radians.

3.

(a) $\sin^2\theta + \cos^2\theta \equiv \left(\frac{v}{R}\right)^2 + \left(\frac{h}{R}\right)^2 \equiv \frac{v^2+h^2}{R^2} \equiv 1$, because R is defined to be $\left(v^2 + h^2\right)^{1/2}$. This result is true regardless of the value of θ; hence we use the identity sigh.

(b) When $\theta = \frac{\pi}{4}$, we have $v = h$, so $R = \sqrt{2v^2} = v\sqrt{2}\left(= h\sqrt{2}\right)$. Thus, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{v}{R} = \frac{v}{v\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

4.

(a) $\sin 2\theta \equiv \sin(\theta+\theta) \equiv \sin\theta\cos\theta + \cos\theta\sin\theta \equiv 2\sin\theta\cos\theta$ [Here,$\theta_1 = \theta_2 = \theta$]

(b) $\cos 2\theta \equiv \cos(\theta+\theta) \equiv \cos\theta\cos\theta - \sin\theta\sin\theta \equiv \cos^2\theta - \sin^2\theta \equiv \cos^2\theta + \sin^2\theta - 2\sin^2\theta \equiv 1 - 2\sin^2\theta$

(c) $\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2) \equiv (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) + (\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) \equiv 2\sin\theta_1\cos\theta_2$

(d) $1 + \tan^2\theta \equiv 1 + \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$

(e) $\sin\left(\frac{\pi}{2} - \theta\right) \equiv \sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta \equiv \cos\theta - 0 \equiv \cos\theta$

(f) $\cos\left(\frac{\pi}{2} - \theta\right) \equiv \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \equiv 0 + \sin\theta \equiv \sin\theta$

5.

(a) $\frac{d}{d\theta}\sin f(\theta) = \frac{d\sin f(\theta)}{df(\theta)}\frac{df(\theta)}{d\theta} = \cos f(\theta)\cdot f'(\theta) = f'(\theta)\cdot\cos f(\theta)$

$\frac{d}{d\theta}\cos f(\theta) = \frac{d\cos f(\theta)}{df(\theta)}\frac{df(\theta)}{d\theta} = -\sin f(\theta)\cdot f'(\theta) = -f'(\theta)\cdot\sin f(\theta)$

(b) $\frac{d}{d\theta}\cos\theta^3 = -3\theta^2\sin\theta^3$

$\frac{d}{d\theta}\sin\left(\theta^2+3\theta\right) = (2\theta+3)\cos\left(\theta^2+3\theta\right)$

$\frac{d}{d\theta}\cos e^\theta = -e^\theta\sin e^\theta$

$\frac{d}{d\theta}\sin\frac{1}{\theta} = -\frac{1}{\theta^2}\cos\frac{1}{\theta}$

6.

(a) $e^{-i\pi} = \cos\pi - i\sin\pi = -1 - 0 = -1$

(b) $e^{-i\pi/3} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \frac{1}{2}\left(1+\sqrt{3}i\right)$

(c) $e^{-i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1+i) = \frac{\sqrt{2}}{2}(1+i)$

(d) $e^{-3i\pi/4} = \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}(1+i) = -\frac{\sqrt{2}}{2}(1+i)$

7.

(a) With $R = 2$ and $\theta = \frac{\pi}{6}$, we find $h = 2\cos\frac{\pi}{6} = \sqrt{3}$ and $v = 2\sin\frac{\pi}{6} = 1$. The Cartesian form is $\sqrt{3} + i$

(b) With $R = 4$ and $\theta = \frac{\pi}{3}$, we find $h = 4\cos\frac{\pi}{3} = 2$ and $v = 4\sin\frac{\pi}{3} = 2\sqrt{3}$. The Cartesian form is $2 + 2\sqrt{3}i$.

(c) Taking the complex number as $\sqrt{2}e^{+i\theta}$, we have $R = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$. So $h = \sqrt{2}\cos\frac{-\pi}{4} = \sqrt{2}\cos\frac{\pi}{4}$ [by (16.14)] $= 1$, and $v = \sqrt{2}\sin\frac{-\pi}{4} = \sqrt{2}\left(-\sin\frac{\pi}{4}\right) = -1$. The Cartesian form is $h + vi = 1 - i$. Alternatively, taking the number as $\sqrt{2}e^{-i\theta}$, we could have $\theta = \frac{\pi}{4}$ instead, with the result that $h = v = 1$. The Cartesian form is then $h - vi = 1 - i$, the same answer.

8.

(a) With $h = \frac{3}{2}$ and $v = \frac{3\sqrt{3}}{2}$, we find $R = 3$. Since θ must satisfy $\cos\theta = \frac{h}{R} = \frac{1}{2}$ and $\sin\theta = \frac{v}{R} = \frac{\sqrt{3}}{2}$ Table 16.2 gives us $\theta = \frac{\pi}{3}$. Thus, $\frac{3}{2} + \frac{3\sqrt{3}}{2}i = 3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 3e^{i\pi/3}$.

(b) With $h = 4\sqrt{3}$ and $v = 4$, we find $R = 8$. In order that $\cos\theta = \frac{h}{R} = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{v}{R} = \frac{1}{2}$, we must have $\theta = \frac{\pi}{6}$. Hence, $4\left(\sqrt{3} + i\right) = 8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 8e^{i\pi/6}$.

Exercise 16.3

1. $a_1 = -4, a_2 = 8, b = 0$. Thus $y_p = 0$. Since $h = 2$, and $v = 2$, we have $y_c = e^{2t}(A_5\cos 2t + A_6\sin 2t)$. The general solution is the same as y_c, since $y_p = 0$. From this solution, we can find that $y(0) = A_5\cos 0 + A_6\sin 0 = A_5$, and $y'(0) = 2\left(A_5 + A_6\right)$. Since the initial conditions are $y(0) = 3$ and $y'(0) = 7$, we get $A_5 = 3$ and $A_6 = \frac{1}{2}$. The definite solution is therefore

$$y\left(t\right) = e^{2t}(3\cos 2t + \frac{1}{2}\sin 2t)$$

2. $a_1 = 4, a_2 = 8, b = 2$. Thus $y_p = \frac{1}{4}$. Since $h = -2$, and $v = 2$, we have $y_c = e^{-2t}(A_5\cos 2t + A_6\sin 2t)$. The general solution is $y(t) = y_c + y_p$. From this solution, we can find $y(0) = A_5 + 1/4$, and $y'(0) = -2A_5 + 2A_6$, which, along with the initial conditions, imply that $A_5 = 2$ and $A_6 = 4$. Thus the definite solution is

$$y\left(t\right) = e^{-2t}(2\cos 2t + 4\sin 2t) + \frac{1}{4}$$

3. $a_1 = 3, a_2 = 4, b = 12$. Thus $y_p = 3$. Since $h = -\frac{3}{2}$, and $v = \frac{\sqrt{7}}{2}$, we have $y_c = e^{-3t/2}(A_5\cos\frac{\sqrt{7}}{2}t + A_6\sin\frac{\sqrt{7}}{2}t)$. The general solution is $y(t) = y_c + y_p$. From this solution, we can find $y(0) = A_5 + 3$, and $y'(0) = -\frac{3}{2}A_5 + \frac{\sqrt{7}}{2}A_6$, which, along with the initial conditions, imply that $A_5 = -1$ and $A_6 = \frac{\sqrt{7}}{7}$. Thus the definite solution is

$$y(t) = e^{-3t/2}(-\cos\frac{\sqrt{7}}{2}t + \frac{\sqrt{7}}{7}\sin\frac{\sqrt{7}}{2}t) + 3$$

4. $a_1 = -2, a_2 = 10, b = 5$. Thus $y_p = \frac{1}{2}$. Since $h = 1$, and $v = 3$, we have $y_c = e^t(A_5 \cos 3t + A_6 \sin 3t)$. The general solution is $y(t) = y_c + y_p$. From this solution, we can find $y(0) = A_5 + \frac{1}{2}$, and $y'(0) = A_5 + 3A_6$, which, in view of the initial conditions, imply that $A_5 = 5\frac{1}{2}$ and $A_6 = 1$. Thus the definite solution is

$$y(t) = e^t(5\frac{1}{2}\cos 3t + \sin 3t) + \frac{1}{2}$$

5. $a_1 = 0, a_2 = 9, b = 3$. Thus $y_p = \frac{1}{2}$. Since $h = 0$, and $v = 3$, and thus $y_c = A_5 \cos 3t + A_6 \sin 3t$. The general solution is $y(t) = y_c + y_p$. From this solution, we can find $y(0) = A_5 + 1/3$, and $y'(0) = 3A_6$, which, by the initial conditions, imply that $A_5 = 2/3$ and $A_6 = 1$. Thus the definite solution is

$$y(t) = \frac{2}{3}\cos 3t + \sin 3t + \frac{1}{3}$$

6. After normalizing (dividing by 2), we have $a_1 = -6, a_2 = 10, b = 20$. Thus $y_p = 2$. Since $h = 3$, and $v = 1$, we have $y_c = e^{3t}(A_5 \cos t + A_6 \sin t)$. The general solution is $y(t) = y_c + y_p$. This solution yields $y(0) = A_5 + 2$, and $y'(0) = 3A_5 + A_6$, which, by the initial conditions, imply that $A_5 = 2$ and $A_6 = -1$. Thus the definite solution is

$$y(t) = e^{3t}(2\cos t - \sin t) + 2$$

7. (a) 2 and 3 (b) 5 (c) 1, 4 and 6

Exercise 16.4

1.

(a) Equating Q_d and Q_s, and normalizing, we have

$$P" + \frac{m-u}{n-w}P' - \frac{\beta+\delta}{n-w}P = -\frac{\alpha+\gamma}{n-w} \qquad (n \neq w)$$

(b) $P_p = \frac{\alpha+\gamma}{\beta+\delta}$

(c) Periodic fluctuation will be absent if

$$\left(\frac{m-u}{n-w}\right)^2 \geq \frac{-4(\beta+\delta)}{n-w}$$

2.

(a) Substitution of Q_d and Q_s, into the market adjustment equation yields (upon normalization)

$$P'' + \frac{jm-1}{jn}P' - \frac{\beta+\delta}{n}P = -\frac{\alpha+\gamma}{n}$$

(b) $\bar{P} = P^* = \frac{\alpha+\gamma}{\beta+\delta}$.

(c) Fluctuation will occur if

$$\left(\frac{jm-1}{jn}\right)^2 < \frac{-4(\beta+\delta)}{n}$$

This condition cannot be satisfied if $n > 0$, for then the right-side expression will be negative, and the square of a real number can never be less than a negative number.

(d) For case 3, dynamic stability requires that

$$h = -\frac{1}{2}\left(\frac{jm-1}{jn}\right) < 0$$

Since $n < 0$ for Case 3, this condition reduces to

$$jm - 1 < 0$$

3.

(a) Equating Q_d and Q_s, and normalizing, we get

$$P\text{"} + P' - \frac{5}{2}P = 5$$

The particular integral is $P_p = 2$. The characteristic roots are complex, with $h = \frac{1}{2}$ and $v = \frac{3}{2}$. Thus the general solution is $P(t) = e^{t/2}\left(A_5 \cos\frac{3}{2}t + A_6 \sin\frac{3}{2}t\right) + 2$. This can be definitized to

$$P(t) = e^{t/2}\left(2\cos\frac{3}{2}t + 2\sin\frac{3}{2}t\right) + 2$$

(b) The path is nonconvergent, and has explosive fluctuation.

Exercise 16.5

1.

(a) Substituting (16.33) into (16.34) yields a first-order differential equation in π:

$$\frac{d\pi}{dt} + j(l-g)\pi = j(\alpha - T - \beta U)$$

(b) A first-order differential equation has only one characteristic root. Since fluctuation is produced by complex roots which come only in conjugate pairs, no fluctuation is now possible.

2. Differentiating (16.33) and (16.35), we get

$$\begin{aligned}\frac{dp}{dt} &= -\beta\frac{dU}{dt} + g\frac{d\pi}{dt} \\ \frac{d^2U}{dt^2} &= k\frac{dp}{dt}\end{aligned}$$

Substitution yields

$$\frac{d^2U}{dt^2} = -k\beta\frac{dU}{dt} + kg\frac{d\pi}{dt} = -k\beta\frac{dU}{dt} + kgj(p-\pi) \quad \text{[by (16.34)]}$$

To get rid of p and π, we note that (16.35) implies

$$p = \frac{1}{k}\frac{dU}{dt} + m$$

and (16.33) implies

$$\pi = \frac{p}{g} - \frac{1}{g}(\alpha - T - \beta U) = \frac{1}{g}\left(\frac{1}{k}\frac{dU}{dt} + m\right) - \frac{1}{g}(\alpha - T - \beta U)$$

Using these to eliminate p and π, and rearranging, we then get the desired differential equation in U:

$$\frac{d^2U}{dt^2} + [k\beta + j(1-h)]\frac{dU}{dt} + (kj\beta)U = kj[\alpha - T - (1-h)m]$$

3.

(a) Under the assumption, (16.33) can be solved for p, to yield

$$p = \frac{1}{1-g}(\alpha - T - \beta U)$$

This gives the derivative

$$\frac{dp}{dt} = -\frac{\beta}{1-g}\frac{dU}{dt} = \frac{\beta km}{1-g} - \frac{\beta k}{1-g}p \quad \text{[by (16.35)]}$$

Thus we have the differential equation

$$\frac{dp}{dt} + \frac{\beta k}{1-g}p = \frac{\beta km}{1-g}$$

(b) Substituting the p expression derived in (a) into (16.35), we obtain (upon rearranging)

$$\frac{dU}{dt} + \frac{k\beta}{1-g}U = -km + \frac{k}{1-g}(\alpha - T)$$

(c) These are first-order differential equations.

(d) It is necessary to have the restriction g$\neq$ 1.

4.

(a) The parameter values are $\beta = 3, g = \frac{1}{3}, j = \frac{3}{4}$ and $k = \frac{1}{2}$. So, with reference to (16.37"), we have

$$a_1 = 2 \quad a_2 = \frac{9}{8} \quad \text{and} \quad b = \frac{9}{8}m$$

The particular integral is $b/a_2 = m$. The characteristic roots are complex, with $h = -1$ and $v = \frac{\sqrt{2}}{4}$. Thus the general solution for π is

$$\pi(t) = e^{-t}\left(A_5 \cos \frac{\sqrt{2}}{4}t + A_6 \sin \frac{\sqrt{2}}{4}t\right) + m$$

Substituting this solution and its derivative into (16.41), and solving for p, we get

$$p(t) = \frac{1}{3}e^{-t}[\left(\sqrt{2}A_6 - A_5\right)\cos \frac{\sqrt{2}}{4}t - \left(\sqrt{2}A_5 + A_6\right)\sin \frac{\sqrt{2}}{4}t] + m$$

The new version of (16.40) implies that $U(t) = \frac{1}{9}\pi - \frac{1}{3}p + \frac{1}{18}$.

Thus

$$U(t) = \frac{1}{9}e^{-t}[\left(2A_5 - \sqrt{2}A_6\right)\cos \frac{\sqrt{2}}{4}t + \left(\sqrt{2}A_5 + 2A_6\right)\sin \frac{\sqrt{2}}{4}t] + \frac{1}{18} - \frac{2}{9}m$$

(b) Yes; yes.

(c) $\overline{p} = m$; $\overline{U} = \frac{1}{18} - \frac{2}{9}m$

(d) Now $\overline{U}$ is functionally related to $\overline{p}$. The long-run Phillips curve is no longer vertical, but negatively sloped. The assumption $g = 1$ (the entire expected rate of inflation is built into the actual rate of inflation) is crucial for the vertical long-run Phillips curve.

Exercise 16.6

1. Given $y"(t) + ay'(t) + by = t^{-1}$, the variable term t^{-1} has successive derivatives involving $t^{-2}, t^{-3}, \ldots$, and giving an infinite number of forms. If we let

$$y(t) = B_1t^{-1} + B_2t^{-2} + B_3t^{-3} + B_4t^{-4} + \ldots$$

There is no end to the $y(t)$ expression. Thus we cannot use it as the particular integral.

2.

(a) Try y_p in the form of $y = B_1t + B_2$. Then $y'(t) = B_1$ and $y"(t) = 0$. Substitution yields $B_1t + (2B_1 + B_2) = t$, thus $B_1 = 1$; moreover, $2B_1 + B_2 = 0$, thus $B_2 = -2$. Hence, $y_p = t - 2$.

(b) Try y_p in the form of $y = B_1t^2 + B_2t + B_3$. Then we have $y'(t) = 2B_1t + B_2$, and $y"(t) = 2B_1$. Substitution now yields $B_1t^2 + (8B_1 + B_2)t + (2B_1 + 4B_2 + B_3) = 2t^2$; Thus $B_1 = 2$, $B_2 = -16$, and $B_3 = 60$. Hence, $y_p = 2t^2 - 16t + 60$.

(c) Try y_p in the form of $y = Be^t$. Then $y'(t) = y"(t) = Be^t$. Substitution yields $4Be^t = e^t$; thus $B = \frac{1}{4}$. Hence, $y_p = \frac{1}{4}e^t$.

(d) Try y_p in the form of $y = B_1 \sin t + B_2 \cos t$. Then we have $y'(t) = B_1 \cos t - B_2 \sin t$, and $y"(t) = -B_1 \sin t - B_2 \cos t$. Substitution yields $(2B_1 - B_2)\sin t + (B_1 + 2B_2)\cos t = \sin t$; Thus $B_1 = \frac{2}{5}$, and $B_2 = -\frac{1}{5}$. Hence, $y_p = \frac{2}{5}\sin t - \frac{1}{5}\cos t$.

Exercise 16.7

1.

(a) Since $a_n \neq 0$, we have $y_p = b/a_n = 8/2 = 4$.

(b) Since $a_n = 0$, but $a_{n-1} \neq 0$, we get $y_p = bt/a_{n-1} = \frac{t}{3}$.

(c) $a_n = a_{n-1} = 0$, but $a_{n-2} \neq 0$. We try the solution $y = kt^2$, so that $y\prime(t) = 2kt$, $y"(t) = 2k$, and $y'''(t) = 0$. Substitution yields $18k = 1$, or $k = 1/18$. Hence, $y_p = \frac{1}{18}t^2$.

(d) We again try $y = kt^2$, so that $y"(t) = 2k$ and $y^{(4)}(t) = 0$. Substitution yields $2k = 4$, or $k = 2$. Hence, $y_p = 2t^2$.

2.

(a) $y_p = 4/2 = 2$. The characteristic roots are real and distinct, with values 1, -1, and 2. Thus the general solution is

$$y(t) = A_1e^t + A_2e^{-t} + A_3e^{2t} + 2$$

(b) $y_p = 0$. The roots are -1, -3, and -3 (repeated). Thus

$$y(t) = A_1e^{-t} + A_2e^{-3t} + A_3te^{-3t}$$

(c) $y_p = 8/8 = 1$. The roots are -4, and $-1 + i$, and $-1 - i$. Thus

$$y(t) = A_1e^{-4t} + e^{-t}(A_2 \cos t + A_3 \sin t) + 1$$

3.

(a) There are two positive roots; the time path is divergent. To use the Routh theorem, we have $a_0 = 1$, $a_1 = -2$, $a_2 = -1$, $a_3 = 2$, and $a_4 = a_5 = 0$. The first determinant is $|a_1| = a_1 = -2 < 0$. Thus the condition for convergence is violated.

(b) All roots are negative; the time path is convergent. Applying the Routh theorem, we have $a_0 = 1$, $a_1 = 7$, $a_2 = 15$, $a_3 = 9$, and $a_4 = a_5 = 0$. The first three determinants have the values of 7, 96 and 864, respectively. Thus convergence is assured.

(c) All roots have negative real parts; the time path is convergent. To use the Routh theorem, we have $a_0 = 1$, $a_1 = 6$, $a_2 = 10$, and $a_3 = 8$. The first three determinants have the values 6, 52 and 416, respectively. Thus convergence is again assured.

4.

(a) Applying the Routh theorem, we have $a_0 = 1$, $a_1 = -10$, $a_2 = 27$, and $a_3 = -18$. The first determinant is $|a_1| = -10 < 0$. Thus the time path must be divergent.

(b) $a_0 = 1$, $a_1 = 11$, $a_2 = 34$, and $a_3 = 24$. The first three determinants are all positive (having the value 11, 350, and 8400, respectively). Hence the path is convergent.

(c) $a_0 = 1$, $a_1 = 4$, $a_2 = 5$, and $a_3 = -2$. The first three determinants have the values 4, 22 and -44, respectively. Since the last determinant is negative, the path is not convergent.

5. The Routh theorem requires that

$$|a_1| = a_1 > 0$$

and $\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} a_1 & 0 \\ 1 & a_2 \end{vmatrix} = a_1 a_2 > 0$

With $a_1 > 0$, this last requirement implies that $a_2 > 0$, too.

CHAPTER 17

Exercise 17.2

1. (a) $y_{t+1} = y_t + 7$ (b) $y_{t+1} = 1.3y_t$ (c) $y_{t+1} = 3y_t - 9$

2.

(a) Iteration yields $y_1 = y_0 + 1$, $y_2 = y_1 + 1 = y_0 + 2$, $y_3 = y_2 + 1 = y_0 + 3$, etc. The solution is $y_t = y_0 + t = 10 + t$

(b) Since $y_1 = \alpha y_0$, $y_2 = \alpha y_1 = \alpha^2 y_0$, $y_3 = \alpha y_2 = \alpha^3 y_0$, etc., the solution is $y_t = \alpha^t y_0 = \beta\alpha^t$

(c) Iteration yields $y_1 = \alpha y_0 - \beta, y_2 = \alpha y_1 - \beta = \alpha^2 y_0 - \alpha\beta - \beta$, $y_3 = \alpha y_2 - \beta = \alpha^3 y_0 - \alpha^2\beta - \alpha\beta - \beta$, etc. The solution is $y_t = \alpha^t y_0 - \beta(\underbrace{\alpha^{t-1} + \alpha^{t-2} + \ldots + \alpha + 1}_{\text{a total of } t \text{ terms}})$

3.

(a) $y_{t+1} - y_t = 1$, so that $a = -1$ and $c = 1$. By (17.9′), the solution is $y_t = y_0 + ct = 10 + t$. The answer checks.

(b) $y_{t+1} - \alpha y_t = 0$, so that $a = -\alpha$, and $c = 0$. Assuming $\alpha \neq 1$, (17.8′) applies, and we have $y_t = y_0\alpha^t = \beta\alpha^t$. It checks. [Assuming $\alpha = 1$ instead, we find from (17.9′) that $y_t = \beta$, which is a special case of $y_t = \beta\alpha^t$.]

(c) $y_{t+1} - \alpha y_t = -\beta$, so that $a = -\alpha$, and $c = -\beta$. Assuming $\alpha \neq 1$, we find from (17.8′) that $y_t = \left(y_0 + \frac{\beta}{1-\alpha}\right)\alpha^t - \frac{\beta}{1-\alpha}$. This is equivalent to the earlier answer, because we can rewrite it as $y_t = y_0\alpha^t - \beta\left(\frac{1-\alpha^t}{1-\alpha}\right) = y_0\alpha^t - \beta(1 + \alpha + \alpha^2 + \ldots + \alpha^{t-1})$.

4.

(a) To find y_c, try the solution $y_t = Ab^t$ in the homogeneous equation $y_{t+1} + 3y_t = 0$. The result is $Ab^{t+1} + 3Ab^t = 0$; i.e., $b = -3$. Hence $y_c = Ab^t = A(-3)^t$. To find y_p, try the solution $y_t = k$ in the complete equation, to get $k + 3k = 4$; i.e. $k = 1$. Hence $y_p = 1$. The general solution is $y_t = A(-3)^t + 1$. Setting $t = 0$ in this solution, we get $y_0 = A + 1$. The initial condition then gives us $A = 3$. The definite solution is $y_t = 3(-3)^t + 1$.

(b) After normalizing the equation to $y_{t+1} - (\frac{1}{2})y_t = 3$, we can find $y_c = Ab^t = A(\frac{1}{2})^t$, and $y_p = k = 6$. Thus, $y_t = A(\frac{1}{2})^t + 6$. Using the initial condition, we get $A = 1$. The definite solution is $y_t = (\frac{1}{2})^t + 6$.

(c) After rewriting the equation as $y_{t+1} - 0.2y_t = 4$, we can find $y_c = A(0.2)^t$, and $y_p = 5$. Thus $y_t = A(0.2)^t + 5$. Using the initial condition, this solution can be definitized to $y_t = -(0.2)^t + 5$.

Exercise 17.3

1. (a) Nonoscillatory; divergent.
   (b) Nonoscillatory; convergent (to zero).
   (c) Oscillatory; convergent.
   (d) Nonoscillatory; convergent.

2. (a) From the expression $3(-3)^t$, we have $b = -3$ (region VII). Thus the path will oscillate explosively around $y_p = 1$.
   (b) With $b = \frac{1}{2}$ (region III), the path will show a nonoscillatory movement from 7 toward $y_p = 6$.
   (c) With $b = 0.2$ (region III again), we have another convergent, nonoscillatory path. But this time it goes upward from an initial value of 1 toward $y_p = 5$.

3. (a) $a = -\frac{1}{3}$, $c = 6$, $y_0 = 1$. By (17.8′), we have $y_t = -8(\frac{1}{3})^t + 9$ — nonoscillatory and convergent.
   (b) $a = 2$, $c = 9$, $y_0 = 4$. By (17.8′), we have $y_t = (-2)^t + 3$ — oscillatory and divergent.
   (c) $a = \frac{1}{4}$, $c = 5$, $y_0 = 2$. By (17.8′), we have $y_t = -2(-\frac{1}{4})^t + 4$ — oscillatory and convergent.
   (d) $a = -1$, $c = 3$, $y_0 = 5$. By (17.9′), we have $y_t = 5 + 3t$ — nonoscillatory and divergent (from a moving equilibrium $3t$).

Exercise 17.4

1. Substitution of the time path (17.12′) into the demand equation leads to the time path of Q_{dt}, which we can simply write as Q_t (since $Q_{dt} = Q_{st}$ by the equilibrium condition:

$$Q_t = \alpha - \beta P_t = \alpha - \beta \left(P_0 - \bar{P}\right) \left(-\frac{\delta}{\beta}\right)^t - \beta \bar{P}$$

Whether Q_t converges depends on the $\left(-\frac{\delta}{\beta}\right)^t$ term, which determines the convergence of P_t as well. Thus P_t and Q_t must be either both convergent, or both divergent.

2. The cobweb in this case will follow a specific rectangular path.

3. (a) $\alpha = 18$, $\beta = 3$, $\gamma = 3$, $\delta = 4$. Thus $\bar{P} = \frac{21}{7} = 3$. Since $\delta > \beta$, there is explosive oscillation.

   (b) $\alpha = 22$, $\beta = 3$, $\gamma = 2$, $\delta = 1$. Thus $\bar{P} = \frac{24}{4} = 6$. Since $\delta < \beta$, the oscillation is damped.

   (c) $\alpha = 19$, $\beta = 6$, $\gamma = 5$, $\delta = 6$. Thus $\bar{P} = \frac{24}{12} = 2$. Since $\delta = \beta$, there is uniform oscillation.

4. (a) The interpretation is that if actual price P_{t-1} exceeds (falls short of) the expected price P^*_{t-1}, then P^*_{t-1} will be revised upward (downward) by a fraction of the discrepancy $P_{t-1} - P^*_{t-1}$, to form the expected price of the next period, P^*_t. The adjustment process is essentially the same as in (16.34), except that, here, time is discrete, and the variable is price rather than the rate of inflation.

   (b) If $\eta = 1$, then $P^*_t = P_{t-1}$ and the model reduces to the cobweb model (17.10). Thus the present model includes the cobweb model as a special case.

   (c) The supply function gives $P^*_t = \frac{Q_{st}+\gamma}{\delta}$, which implies that $P^*_{t-1} = \frac{Q_{s,t-1}+\gamma}{\delta}$. But since $Q_{st} = Q_{dt} = \alpha - \beta P_t$, and similarly, $Q_{s,t-1} = \alpha - \beta P_{t-1}$, we have

$$P^*_t = \frac{\alpha + \gamma - \beta P_t}{\delta} \quad \text{and} \quad P^*_{t-1} = \frac{\alpha + \gamma - \beta P_{t-1}}{\delta}$$

   Substituting these into the adaptive expectations equation, and simplifying and shifting the time subscript by one period, we obtain the equation

$$P_{t+1} - \left(1 - \eta - \frac{\eta\delta}{\beta}\right) P_t = \frac{\eta(\alpha+\gamma)}{\beta}$$

   which is in the form of (17.6) with $a = -\left(1 - \eta - \frac{\eta\delta}{\beta}\right) \neq -1$, and $c = \frac{\eta(\alpha+\gamma)}{\beta}$.

(d) Since $a \neq -1$, we can apply formula (17.8′) to get

$$\begin{aligned} P_t &= \left(P_0 - \frac{\alpha+\gamma}{\beta+\delta}\right)\left(1-\eta-\frac{\eta\delta}{\beta}\right)^t + \frac{\alpha+\gamma}{\beta+\delta} \\ &= (P_0 - \bar{P})\left(1-\eta-\frac{\eta\delta}{\beta}\right)^t + \bar{P} \end{aligned}$$

This time path is not necessarily oscillatory, but it will be if $\left(1-\eta-\frac{\eta\delta}{\beta}\right)$ is negative, i.e., if $\frac{\beta}{\beta+\delta} < \eta$.

(e) If the price path is oscillatory and convergent (region V in Fig. 17.1), we must have $-1 < 1-\eta-\frac{\eta\delta}{\beta} < 0$, where the second inequality has to do with the presence of oscillation, and the first, with the question of convergence. Adding $(\eta - 1)$, and dividing throught by η, we have $1-\frac{2}{\eta} < -\frac{\delta}{\beta} < 1-\frac{1}{\eta}$. Given that the path is oscillatory, convergence requires $1-\frac{2}{\eta} < -\frac{\delta}{\beta}$. If $\eta = 1$ (cobweb model), the stability-inducing range for $-\frac{\delta}{\beta}$ is $-1 < -\frac{\delta}{\beta} < 0$. If $0 < \eta < 1$, however, the range will become wider. With $\eta = \frac{1}{2}$, e.g., the range becomes $-3 < -\frac{\delta}{\beta} < -1$.

5. The dynamizing agent is the lag in the supply function. This introduces P_{t-1} into the model, which together with P_t, forms a pattern of change.

Exercise 17.5

1. Because $a = \sigma(\beta+\delta) - 1 \neq -1$, by model specification.

2. (IV) $1 - \sigma(\beta+\delta) = 0$. Thus $\sigma = \frac{1}{\beta+\delta}$.

   (V) $-1 < 1-\sigma(\beta+\delta) < 0$. Subtracting 1, we get $-2 < -\sigma(\beta+\delta) < -1$. Multiplying by $\frac{-1}{\beta+\delta}$, we obtain $\frac{2}{\beta+\delta} > \sigma > \frac{1}{\beta+\delta}$.

   (VI) $1-\sigma(\beta+\delta) = -1$. Thus $\sigma = \frac{2}{\beta+\delta}$.

   (VII) $1-\sigma(\beta+\delta) < -1$. Subtracting 1, and multiplying by $\frac{-1}{\beta+\delta}$, we obtain $\sigma > \frac{2}{\beta+\delta}$.

3. With $\sigma = 0.3$, $\alpha = 21$, $\beta = 2$, $\gamma = 3$, and $\delta = 6$, we find from (16.15) that $P_t = (P_0 - 3)(-1.4)^t + 3$, a case of explosive oscillation.

4. The difference equation will become $P_{t+1} - (1-\sigma\beta)P_t = \sigma(\alpha - k)$, with solution

$$P_t = \left(P_0 - \frac{\alpha-k}{\beta}\right)(1-\alpha\beta)^t + \frac{\alpha-k}{\beta}$$

The term $b_t = (1 - \sigma\beta)^t$ is decisive in the time-path configuration:

Region	b	σ
III	$0 < b < 1$	$0 < \sigma < \frac{1}{\beta}$
IV	$b = 0$	$\sigma = \frac{1}{\beta}$
V	$-1 < b < 0$	$\frac{1}{\beta} < \sigma < \frac{2}{\beta}$
VI	$b = -1$	$\sigma = \frac{2}{\beta}$
VII	$b < -1$	$\sigma > \frac{2}{\beta}$

[These results are the same as Table 17.2 with δ set equal to 0.] To have a positive $\bar{P}$, we must ahve $k < \alpha$; that is, the horizontal supply curve must be located below the vertical intercept of the demand curve.

Exercise 17.6

1. No, y_t and y_{t+1} can take any real values, and are continuous.

2. (a) Yes, L and R give two equilibria.
   (b) Nonoscillatory, explosive downward movement.
   (c) Damped, steady upward movement toward R.
   (d) Damped, steady downward movement toward R.
   (e) L is an unstable equilibrium; R is a stable one.

3. (a) Yes.
   (b) Nonoscillatory explosive decrease.
   (c) At first ther iwll be steady downward movement to the right, but as it approaches R, oscillation will develop because of the negative slope of the phase line. Whether the oscillation will be explosive depends on the steepness of the negatively-sloped segment of the curve.
   (d) Oscillation around R will again occur — either explosive, or damped, provided y_0 maps to a point on the phase line higher than L.
   (e) L is definitely unstable. The stability of R depends on the steepness of the curve.

4. (a) The phase line will be downward-sloping at first, but will become horizontal at the level of P_m on the vertical axis.

(b) Yes; yes.

(c) Yes.

5. From equation (17.17), we can write for the kink point:

$$\hat{P} = \frac{\alpha + \gamma}{\beta} - \frac{\delta}{\beta}k \quad \text{or} \quad \frac{\delta}{\beta}k = \frac{\alpha + \gamma}{\beta} - \hat{P}$$

It follows that

$$k = \frac{\beta}{\delta}\left(\frac{\alpha + \gamma}{\beta} - \hat{P}\right) = \frac{\alpha + \gamma}{\beta} - \frac{\beta}{\delta}\hat{P}$$

CHAPTER 18

Exercise 18.1

1. (a) $b^2 - b + \frac{1}{2} = 0$; $b_1, b_2 = \frac{1}{2}(1 \pm \sqrt{1-2}) = \frac{1}{2} \pm \frac{1}{2}i$.

   (b) $b^2 - 4b + 4 = 0$; $b_1, b_2 = \frac{1}{2}(4 \pm \sqrt{16-16}) = 2, 2$.

   (c) $b^2 + \frac{1}{2}b - \frac{1}{2} = 0$; $b_1, b_2 = \frac{1}{2}(-\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{8}{4}}) = \frac{1}{2}, -1$.

   (d) $b^2 - 2b + 3 = 0$; $b_1, b_2 = \frac{1}{2}(2 \pm \sqrt{4-12}) = 1 \pm \sqrt{2}i$.

2. (a) Complex roots imply stepped fluctuation. Since the absolute value of the roots is $R = \sqrt{a_2} = \sqrt{1/2} < 1$, it is damped.

   (b) With repeated roots greater than one, the path is nonoscillatory and explosive.

   (c) The roots are real and distinct; -1 is the dominant root. It's negativity implies oscillation, and its unit absolute value implies that oscillation will eventually become uniform.

   (d) The complex roots have an absolute value greater that 1: $R = \sqrt{3}$. Thus there is explosive stepped fluctuation.

3. (a) $a_1 = -1$, $a_2 = \frac{1}{2}$, and $c = 2$. By (18.2), $y_p = \frac{2}{1/2} = 4$.

   (b) $y_p = 7/1 = 7$ (c) $y_p = 5/1 = 5$ (d) $y_p = 4/2 = 2$

   All of these represent stationary equilibria.

4. (a) $a_1 = 3$, $a_2 = -7/4$, and $c = 9$. $y_p = \frac{9}{1+3-7/4} = 4$. With characteristic roots $b_1, b_2 = \frac{1}{2}(-3 \pm \sqrt{9+7}) = \frac{1}{2}, -\frac{7}{2}$, the general solution is: $y_t = A_1(\frac{1}{2})^t + A_2(-\frac{7}{2})^t + 4$. Setting $t = 0$ in this solution, and using the initial condition $y_0 = 6$, we have $6 = A_1 + A_2 = 4$; Thus $A_1 + A_2 = 2$. Next, setting $t = 1$, and using $y_1 = 3$, we have $3 = \frac{1}{2}A_1 - \frac{7}{2}A_2 = -1$. These results give us $A_1 = 3/2$ and $A_2 = 1/2$. Therefore, the definite solution is

$$y_t = \frac{3}{2}\left(\frac{1}{2}\right)^t + \frac{1}{2}\left(-\frac{7}{2}\right)^t + 4$$

(b) $a_1 = -2$, $a_2 = 2$, and $c = 1$. $y_p = \frac{1}{1-2+2} = 1$. The roots are $b_1, b_2 = \frac{1}{2}(2 \pm \sqrt{4-8}) = 1 \pm i$, giving us $h = v = 1$. Since $R = \sqrt{2}$, we find from (18.9) and Table 16.2 that $\theta = \pi/4$. The general solution is: $y_t = (\sqrt{2})^t(A_5 \cos\frac{\pi}{4}t + A_6 \sin\frac{\pi}{4}t) + 1$. Setting $t = 0$, and using the condition $y_0 = 3$, we obtain $3 = (A_5 \cos 0 + A_6 \sin 0) + 1 = A_5 + 0 + 1$; thus $A_5 = 2$. Next, setting $t = 1$, and using $y_1 = 4$, we find $4 = \sqrt{2}(2\cos\frac{\pi}{4} + A_6 \sin\frac{\pi}{4}) + 1 = \sqrt{2}(\frac{2}{\sqrt{2}} + \frac{A_6}{\sqrt{2}}) + 1 = 2 + A_6 + 1$; thus $A_6 = 1$. The definite solution is therefore

$$y_t = (\sqrt{2})^t\left(2\cos\frac{\pi}{4}t + \sin\frac{\pi}{4}t\right) + 1$$

(c) $a_1 = -1$, $a_2 = 1/4$, and $c = 2$. $y_p = \frac{2}{1-1+1/4} = 8$. With roots $b_1, b_2 = \frac{1}{2}(1 \pm \sqrt{1-1}) = 1/2, 1/2$ (repeated), the general solution is $y_t = A_3(1/2)^t + A_4 t(1/2)^t + 8$. Using the initial conditions, we find $A_3 = -4$ and $A_4 = 2$. Thus the definite solution is

$$y_t = -4\left(\frac{1}{2}\right)^t + 2t\left(\frac{1}{2}\right)^t + 8$$

5. (a) The dominant root being $-7/2$, the time path will eventually be characterized by explosive oscillation.

   (b) The complex roots imply stepped fluctuation. Since $R = \sqrt{2} > 1$, the fluctuation is explosive.

   (c) The repeated roots lie between 0 and 1; the time path is thus nonoscillatory and convergent.

Exercise 18.2

1. (a) Subcase 1D (b) Subcase 3D

   (c) Subcase 1C (d) Subcase 3C

2. (a) $b_1, b_2 = \frac{1}{2}\left[\gamma(1+\alpha) \pm \sqrt{\gamma^2(1+\alpha)^2 - 4\alpha\gamma}\right] = \frac{1}{2}(3.6 \pm \sqrt{1.76}) \approx 2.46, 1.13$. The path should be divergent.

   (b) $b_1, b_2 = \frac{1}{2}(1.08 \pm \sqrt{0.466}) \approx 0.87, 0.21$. The path should be convergent.

3. For Possibilities ii and iv, with either $b_1 = 1$, or $b_2 = 1$, we find $(1-b_1)(1-b_2) = 0$. Thus,

by (18.16), $1-\gamma = 0$, or $\gamma = 1$. for Possibility iii, with $b_1 > 1$ and b_2 a positive fraction, $(1-b_1)(1-b_2)$ is negative. Thus, by (18.16), $1-\gamma < 0$, or $\gamma > 1$.

4. Case 3 is characterized by $\gamma < \frac{4\alpha}{(1+\alpha)^2}$. If $\gamma \geq 1$, then it follows that $1 < \frac{4\alpha}{(1+\alpha)^2}$. Multiplying through by $(1+\alpha)^2$, and subtracting 4α from both sides, we get $1-2\alpha+\alpha^2 < 0$, which can be written as $(1-\alpha)^2 < 0$. But this inequality is impossible, since the square of a real number can never be negative. Hence we cannot have $\gamma \geq 1$ in Case 3.

Exercise 18.3

1. (a) Shifting the time subscripts in (18.23) forward one period, we get

$$(1+\beta k)\, p_{t+2} - [1-j(1-g)]\, p_{t+1} + j\beta U_{t+1} = \beta km + j(\alpha - T)$$

(b) Subtracting (18.23) from the above result, we have

$$(1+\beta k)\, p_{t+2} - [2+\beta k - j(1-g)]\, p_{t+1} + [1-j(1-g)]\, p_t + j\beta(U_{t+1} - U_t) = 0$$

(c) Now we substitute (18.20) to obtain

$$(1+\beta k)\, p_{t+2} - [1+gj+(1-j)(1+\beta k)]\, p_{t+1} + [1-j(1-g)]\, p_t = j\beta km$$

(d) When we divide through by $(1+\beta k)$, the result is (18.24).

2. Substituting (18.18) into (18.19) and collecting terms, we get

$$\pi_{t+1} - (1-j+jg)\pi_t = j(\alpha - T) - j\beta U_t$$

Differencing this result yields [by (18.20)]

$$\begin{aligned}\pi_{t+2} - (2-j+jg)\pi_{t+1} + (1-j+jg)\pi_t &= -j\beta(U_{t+1} - U_t) \\ &= j\beta km - j\beta p_{t+1}\end{aligned}$$

A forward shifted version of (18.19) gives us $jp_{t+1} = \pi_{t+2} - (1-j)\pi_{t+1}$. Using this to eliminate the p_{t+1} term in the preceding result, we get

$$(1+\beta k)\pi_{t+2} - [1 + jg + (1-j)(1-k\beta)]\,\pi_{t+1} + (1 - j + jg)\pi_t = j\beta km$$

When normalized, this becomes a difference equation with the same constant coefficients and constant terms as in (18.24).

3. Let $g > 1$. Then from (18.26) and (18.27), we still have $b_1 + b_2 > 0$ and $(1-b_1)(1-b_2) > 0$. But in (18.26′) we note that $b_1 b_2$ can now exceed one. This would make feasible Possibility v (Case 1), Possibility viii (Case 2), and Possibilities x and xi (Case 3), all of which imply divergence.

4. (a) The first line of (18.21) is still valid, but its second line now becomes

$$P_{t+1} - p_t = \beta k(m - p_t) + gj(p_t - \pi_t)$$

Consequently, (18.23) becomes

$$P_{t+1} - [1 - j(1-g) - \beta k]\,p_t + j\beta U_t = \beta km + j(\alpha - T)$$

And (18.24) becomes

$$P_{t+1} - [2 - j(1-g) - \beta k]\,p_{t+1} + [1 - j(1-g) - \beta k(1-j)]p_t = j\beta km$$

(b) No, we still have $\bar{p} = m$.

(c) With $j = g = 1$, we have $a_1 = \beta k - 2$ and $a_2 = 1$. Thus $a_1^2 \gtreqless 4a_2$ iff $(\beta k - 2) \gtreqless 4$ iff $\beta k \gtreqless 4$. The value of βk marks off the three cases from one another.

(d) With $\beta k = 3$, the roots are complex, with $R = \sqrt{a_2} = 1$; the path has stepped fluctuation and is nonconvergent. With $\beta k = 4$, we have repeated roots, with $b = -\frac{1}{2}(4-2) = -1$; the time path has nonconvergent oscillation. With $\beta k = 5$, we have distinct real roots, $b_1, b_2 = \frac{1}{2}(-3 \pm \sqrt{5}) = -0.38, -2.62$; the time path has divergent oscillation.

Exercise 18.4

1. (a) $\Delta t = (t+1) - t = 1$ (b) $\Delta^2 t = \Delta(\Delta t) = \Delta(1) = 0$

These results are similar to $\frac{d}{dt}t = 1$ and $\frac{d^2}{dt^2}t = 0$.

(c) $\Delta t^3 = (t+1)^3 - t^3 = 3t^2 + 3t + 1$.

This result is very much different from $\frac{d}{dt}t^3 = 3t^2$.

2. (a) $c = 1$, $m = 3$, $a_1 = 2$, and $a_2 = 1$; (17.36) gives $y_p = \frac{1}{16}(3)^t$.

   (b) Formula (18.36) does not apply since $m^2 + a_1 m + a_2 = 0$. We try the solution $y_t = Bt(6)^t$, and obtain the equation $B(t+2)(6)^{t+2} - 5B(t+1)(6)^{t+1} - 6Bt(6)^t = 2(6)^t$. This reduces to $42B = 2$. Thus $B = 1/21$ and $y_p = \frac{1}{21}t(6)^t$.

   (c) After normalization, we find $c = 1$, $m = 4$, $a_1 = 0$, and $a_2 = 3$. By (18.36), we have $y_p = \frac{1}{19}(4)^t$.

3. (a) The trial solution is $y_t = B_0 + B_1 t$, which implies that $y_{t+1} = B_0 + B_1(t+1) = (B_0 + B_1) + B_1 t$, and $y_{t+2} = B_0 + B_1(t+2) = (B_0 + 2B_1) + B_1 t$. Substitution into the difference equation yields $4B_0 + 4B_1 t = t$, so $B_0 = 0$ and $B_1 = 1/4$. Thus $y_p = t/4$.

   (b) This is the same equation as in (a) except for the variable term. With the same trial solution, we get by substitution $4B_0 + 4B_1 t = 4 + 2t$. Thus $B_0 = 1$ and $B_1 = 1/2$, and $y_p = 1 + t/2$.

   (c) The trial solution is $y_t = B_0 + B_1 t + B_2 t^2$ (same as in Example 2). Substituting this (and the corresponding y_{t+1} and y_{t+2} forms) into the equation, we get

$$(8B_0 + 7B_1 + 9B_2) + (8B_1 + 14B_2)t + 8B_2 t^2 = 18 + 6t + 8t^2$$

   Thus $B_0 = 2$, $B_1 = -1$, $B_2 = 1$, and $y_p = 2 - t + t^2$.

4. Upon successive differencing, the m^t part of the variable term gives rise to expressions in the form $B(m)^t$, whereas the t^n part leads to those in the form $(B_0 + \ldots + B_n t^n)$. The trial solution must take both of these into account.

5. (a) The characteristic equation is $b^3 - b^2/2 - b + 1/2 = 0$, which can be written as $(b - 1/2)(b^2 - 1) = (b - 1/2)(b + 1)(b - 1) = 0$. The roots are $1/2, -1, 1$, and we have $y_c = A_1(1/2)^t + A_2(-1)^t + A_3$.

(b) The characteristic equation is $b^3 - 2b^2 - 5b/4 - 1/4 = 0$, which can be written as $(b - 1/2)(b^2 - 3b/2 + 1/2) = 0$. The first factor gives the root 1/2; the second gives the roots $1, 1/2,$. Since the two roots are repeated, we must write $y_c = A_1(1/2)^t + A_2 t(1/2)^t + A_3$.

6. (a) Since $n = 2$, $a_0 = 1$, $a_1 = 1/2$ and $a_2 = -1/2$, we have

$$\Delta_1 = \begin{vmatrix} 1 & -1/2 \\ -1/2 & 1 \end{vmatrix} = \frac{3}{4} > 0, \quad \text{but} \quad \Delta_2 = \begin{vmatrix} 1 & 0 & -1/2 & 1/2 \\ 1/2 & 1 & 0 & -1/2 \\ -1/2 & 0 & 1 & 1/2 \\ 1/2 & -1/2 & 0 & 1 \end{vmatrix} = 0$$

Thus the time path is not convergent.

(b) Since $a_0 = 1$, $a_1 = 0$ and $a_2 = -1/9$, we have

$$\Delta_1 = \begin{vmatrix} 1 & -1/9 \\ -1/9 & 1 \end{vmatrix} = \frac{80}{81}; \Delta_2 = \begin{vmatrix} 1 & 0 & -1/9 & 0 \\ 0 & 1 & 0 & -1/9 \\ -1/9 & 0 & 1 & 0 \\ 0 & -1/9 & 0 & 0 \end{vmatrix} = \frac{6400}{6561}$$

The time path is convergent.

7. Since $n = 3$, there are three determinants as follows:

$$\Delta_1 = \begin{vmatrix} 1 & a_3 \\ a_3 & 1 \end{vmatrix} \qquad \Delta_2 = \begin{vmatrix} 1 & 0 & a_3 & a_2 \\ a_1 & 1 & 0 & a_3 \\ a_3 & 0 & 1 & a_1 \\ a_2 & a_3 & 0 & 1 \end{vmatrix}$$

and

$$\Delta_3 = \begin{vmatrix} 1 & 0 & 0 & a_3 & a_2 & a_1 \\ a_1 & 1 & 0 & 0 & a_3 & a_2 \\ a_2 & a_1 & 1 & 0 & 0 & a_3 \\ a_3 & 0 & 0 & 1 & a_1 & a_2 \\ a_2 & a_3 & 0 & 0 & 1 & a_1 \\ a_1 & a_2 & a_3 & 0 & 0 & 1 \end{vmatrix}$$

CHAPTER 19

Exercise 19.2

1. The equation $y_{t+2} + 6y_{t+1} + 9y_t = 4$ is a specific example of (19.1), with $a_1 = 6$, $a_2 = 9$, and $c = 4$. When these values are inserted into (19.1'), we get precisely the system (19.4). The solution is Example 4 of Sec.18.1 is exactly the same as that for the variable y obtained from the system (19.4), but it does not give the time path for x, since the variable x is absent from the single-equation formulation.

2. The characteristic equation of (19.2) can be written immediately as $b^3 + b^2 - 3b + 2 = 0$. As to (19.2'), the characteristic equation should be $|b\mathrm{I} + K| = 0$; since $K = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$, we have $|b\mathrm{I} + K| = \begin{bmatrix} b+1 & -3 & 2 \\ -1 & b & 0 \\ 0 & -1 & b \end{bmatrix} = b^3 + b^2 - 3b + 2 = 0$ which is exactly the same.

3. (a) To find the particular solution, use (19.5'):

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = (I+K)^{-1}d = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 24 \\ 9 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 24 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

To find the complementary functions, we first form the characteristic equation by using (19.9'): $|b\mathrm{I} + K| = \begin{vmatrix} b+1 & 2 \\ 2 & b-2 \end{vmatrix} = b^2 - b - 6 = 0$ The roots $b_1 = 3$ and $b_2 = -2$ yield the following sets of m and n values: $m_1 = -A_1$, $n_1 = 2A_1$; $m_2 = 2A_2$, $n_2 = A_2$. Thus we have $x_c = -A_1(3)^t + 2A_2(-2)^t)$ and $y_c = 2A_1(3)^t + A_2(-2)^t$. Adding the particular solutions to these complementary functions and definitizing the constants A_i, we finally get the time paths $x_t = -3^t + 4(-2)^t + 7$ and $y_t = 2(3)^t + 2(-2)^t + 5$.

(b) The particular solutions can be found by setting all x's equal to $\overline{x}$ and all y's equal to $\overline{y}$, and solving the resulting equations. The answers are $\overline{x} = 6$, and $\overline{y} = 3$. If the matrix method is used, we must modify (19.5') by replacing I with $J = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Thus

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = (J+K)^{-1}d = \begin{bmatrix} 0 & -\frac{1}{3} \\ 1 & \frac{5}{6} \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 8\frac{1}{2} \end{bmatrix} = \frac{1}{6} \begin{bmatrix} \frac{5}{2} & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 8\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

The characteristic equation,

$|b\mathrm{J}+K| = \begin{vmatrix} b-1 & -\frac{1}{3} \\ b & b-\frac{1}{6} \end{vmatrix} = b^2 - \frac{5}{6}b + \frac{1}{6} = 0$, has roots $b_1 = \frac{1}{2}$ and $b_2 = \frac{1}{3}$. These imply: $m_1 = 2A_1$, $n_1 = -3A_1$; $m_2 = A_2$, $n_2 = -2A_2$. Thus the complementary functions are $x_c = 2A_1(\frac{1}{2})^t + A_2(\frac{1}{3})^t)$ and $y_c = -3A_1(\frac{1}{2})^t - 2A_2(\frac{1}{3})^t$. Combining these with the particular solutions, and definitizing the constants A_i, we finally obtain the time paths $x_t = -2(\frac{1}{2})^t + (\frac{1}{3})^t + 6$ and $y_t = 3(\frac{1}{2})^t - 2(\frac{1}{3})^t + 3$.

4. (a) To find the particular integrals, we utilize (19.14):

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = (M)^{-1}g = \begin{bmatrix} -1 & -12 \\ 1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -60 \\ 36 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 & 12 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -60 \\ 36 \end{bmatrix} = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$$

The characteristic equation, $|r\mathrm{I}+M| = \begin{vmatrix} r-1 & -12 \\ 1 & r+6 \end{vmatrix} = r^2 + 5r + 6 = 0$, has roots $r_1 = -2$ and $r_2 = -3$. These imply: $m_1 = -4A_1$, $n_1 = A_1$; $m_2 = -3A_2$, $n_2 = A_2$. Thus the complementary functions are $x_c = -4A_1e^{-2t} - 3A_2e^{-3t}$ and $y_c = A_1e^{-2t} + A_2e^{-3t}$. Combining these with the particular solutions, and definitizing the constants A_i, we find the time paths to be $x(t) = 4e^{-2t} - 3e^{-3t} + 12$ and $y(t) = -e^{-2t} + e^{-3t} + 4$.

(b) The particular integrals are, according to (19.14),

$$\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = (M)^{-1}g = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 9 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 9 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

The characteristic equation, $|r\mathrm{I}+M| = \begin{vmatrix} r-2 & 3 \\ -1 & r+2 \end{vmatrix} = r^2 - 1 = 0$, has roots $r_1 = 1$ and $r_2 = -1$. These imply: $m_1 = 3A_1$, $n_1 = A_1$; $m_2 = A_2$, $n_2 = A_2$. Thus the complementary functions are $x_c = 3A_1e^t + A_2e^{-t}$ and $y_c = A_1e^t + A_2e^{-t}$. Combining these with the particular solutions, and definitizing the constants A_i, we find the time paths to be $x(t) = 6e^t - 5e^{-t} + 7$ and $y(t) = 2e^t - 5e^{-t} + 8$.

5. The system (19.13) is in the format of $Ju + Mv = g$, and the desired matrix is $D = -J^{-1}M$. Since $J^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ and $M = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$, we have $D = \begin{bmatrix} 0 & 3 \\ -1 & -4 \end{bmatrix}$. The characteristic

equation of this matrix is $|D - rI| = 0$ or $\begin{vmatrix} -r & 3 \\ -1 & -4-r \end{vmatrix} = r^2 + 4r + 3 = 0$, which checks with (19.16').

Exercise 19.3

1. Since $d_t = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \delta^t$, we have $\begin{bmatrix} \delta - a_{11} & -a_{12} \\ -a_{21} & \delta - a_{22} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$. Thus $\beta_1 = \frac{1}{\Delta}[\lambda_1(\delta - a_{22}) + \lambda_2 a_{12}]$ and $\beta_2 = \frac{1}{\Delta}[\lambda_2(\delta - a_{11}) + \lambda_1 a_{21}]$, where $\Delta = (\delta - a_{11})(\delta - a_{22}) - a_{12}a_{21}$. It is clear that the answers in Example 1 are the special case where $\lambda_1 = \lambda_2 = 1$.

2. (a) The key to rewriting process is the fact that $\delta I = \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix}$. The rest follows easily.

   (b) Scalar: δ. Vectors: β, u. Matrices: I, A.

   (c) $\beta = (\delta I - A)^{-1} u$

3. (a) $\rho I + I - A = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \rho + 1 - a_{11} & -a_{12} \\ -a_{21} & \rho + 1 - a_{22} \end{bmatrix}$. The rest follows easily.

   (b) Scalar: ρ. Vectors: β, λ. Matrices: I, A.

   (c) $\beta = (\rho I + I - A)^{-1} \lambda$.

4. (a) With trial solution $\beta_i \delta^t = \beta_i (\frac{10}{12})^t$, we find from (19.22') that $\beta_1 = \frac{70}{39}$ and $\beta_2 = \frac{20}{13}$. So $x_{1p} = \frac{70}{39}(\frac{12}{10})^t$ and $x_{2p} = \frac{20}{13}(\frac{12}{10})^t$.

   (b) From the equation $\begin{vmatrix} b - \frac{3}{10} & -\frac{4}{10} \\ -\frac{3}{10} & b - \frac{2}{10} \end{vmatrix}] = b^2 - \frac{5}{10}b - \frac{6}{100} = 0$, we find $b_1 = \frac{6}{10}$,$b_2 = -\frac{1}{10}$. These give us $m_1 = 4A_1$, $n_1 = 3A_1$; $m_2 = A_2$, $n_2 = -A_2$. Thus $x_{1c} = 4A_1(\frac{6}{10})^t + A_2(-\frac{1}{10})^t$ and $x_{2c} = 3A_1(\frac{6}{10})^t - A_2(-\frac{1}{10})^t$

   (c) Combining the above results, and utilizing the initial conditions, we find $A_1 = 1$ and $A_2 = -1$. Thus the time paths are

$$x_{1,t} = 4(\frac{6}{10})^t - (-\frac{1}{10})^t + \frac{70}{39}(\frac{12}{10})^t$$

$$x_{2,t} = 3(\frac{6}{10})^t - (-\frac{1}{10})^t + \frac{20}{13}(\frac{12}{10})^t$$

5. (a) With trial solution $\beta_i e^{\rho t} = \beta_i e^{\frac{t}{10}}$, we find from (19.25') that $\beta_1 = \frac{17}{6}$ and $\beta_2 = \frac{19}{6}$. So $x_{1p} = \frac{17}{6}e^{t/10}$ and $x_{2p} = \frac{19}{6}e^{t/10}$.

(b) From the equation $\begin{vmatrix} r+1-\frac{3}{10} & -\frac{4}{10} \\ -\frac{3}{10} & r+1-\frac{2}{10} \end{vmatrix}] = r^2 + \frac{15}{10}r + \frac{44}{100} = 0$, we find $r_1 = -\frac{4}{10}$,$r_2 = -\frac{11}{10}$. These give us $m_1 = 4A_1$, $n_1 = 3A_1$; $m_2 = A_2$, $n_2 = -A_2$. Thus $x_{1c} = 4A_1e^{-4t/10} + A_2e^{-11t/10}$ and $x_{2c} = 3A_1e^{-4t/10} - A_2e^{-11t/10}$

(c) Combining the above results, and utilizing the initial conditions, we find $A_1 = 1$ and $A_2 = 2$. Thus the time paths are

$$x_{1,t} = 4e^{-4t/10} + 2e^{-11t/10} + \frac{17}{6}e^{t/10}$$
$$x_{2,t} = 3e^{-4t/10} - 2e^{-11t/10} + \frac{19}{6}e^{t/10}$$

6. (a) E,a and P are $n \times 1$ column vectors; A is an $n \times n$ matrix.

(b) The interpretation is that, at any instant of time, an excess demand for the ith product will induce a price adjustment to the extent of α_i times the magnitude of excess demand.

(c) $\frac{dP_1}{dt} = \alpha_1(a_{10} + a_{11}P_1 + a_{12}P_2 + \ldots + a_{1n}P_n)$

$\vdots$

$\frac{dP_n}{dt} = \alpha_n(a_{n0} + a_{n1}P_1 + a_{n2}P_2 + \ldots + a_{nn}P_n)$

(d) It can be verified that $P' = \alpha E$. Thus we have

$$P' = \alpha(a + AP)$$

or

$$\underset{(n\times1)}{P'} - \underset{(n\times n)}{\alpha}\ \underset{(n\times n)}{A}\ \underset{(n\times1)}{P} = \underset{(n\times n)}{\alpha}\ \underset{(n\times1)}{a}$$

7. (a) $E_{1,t} = a_{10} + a_{11}P_{1,t} + a_{12}P_{2,t} + \ldots + a_{1n}P_{n,t})$

$\vdots$

$E_{n,t} = a_{n0} + a_{n1}P_{1,t} + a_{n2}P_{2,t} + \ldots + a_{nn}P_{n,t})$

Thus we have $E_t = a + AP_t$.

(b) Since $\Delta P_{i,t} \equiv P_{i,t+1} - P_{i,t}$, we can write

$$\begin{bmatrix} \Delta P_{1,t} \\ \vdots \\ \Delta P_{n,t} \end{bmatrix} = \begin{bmatrix} P_{1,t+1} - P_{1,t} \\ \vdots \\ P_{n,t+1} - P_{n,t} \end{bmatrix} = \underset{(n\times1)}{P_{t+1}} - \underset{(n\times1)}{P_t}$$

The rest follows easily.

(c) Inasmuch as $P_{t+1} - P_t = \alpha E_t = \alpha a + \alpha AP_t$ it follows that $P_{t+1} - IP_t - \alpha AP_t = \alpha a$ or

$$P_{t+1} - (I + \alpha A)P_t = \alpha a$$

Exercise 19.4

1. Cramer's rule makes use of the determinants:

   $|A| = \kappa\beta j \quad |A_1| = \kappa\beta j\mu \quad |A_2| = \kappa j(\alpha - T - \mu(1-g))$

   Then we have :

   $\bar{\pi} = \frac{|A_1|}{|A|} = \mu, \quad \bar{U} = \frac{|A_2|}{|A|} = \frac{\alpha - T - \mu(1-g)}{\beta}$

2. The first equation in (19.34) gives us

$$-\frac{3}{4}(1-i)m_1 = -\frac{9}{4}n_1$$

   Multiplying through by $-\frac{4}{9}$ we get

$$\frac{1}{3}(1-i)m_1 = n_1$$

   The second equation in (19.34) gives us

$$-\frac{1}{2}m_1 = -\frac{3}{4}(1+i)n_1$$

   Multiplying through by $-\frac{2}{3}(1-i)$, and noting that $(1+i)(1-i) = 1 - i^2 = 2$, we again get

$$\frac{1}{3}(1-i)m_1 = n_1$$

3. With $\alpha - t = \frac{1}{6}$, $\beta = 2$, $h = 1/3$, $j = 1/4$ and $\kappa = 1/2$, the system (19.28') becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \pi' \\ U' \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & \frac{1}{2} \\ -\frac{1}{6} & 1 \end{bmatrix} \begin{bmatrix} \pi \\ U \end{bmatrix} = \begin{bmatrix} \frac{1}{24} \\ \frac{1}{12} - \frac{\mu}{2} \end{bmatrix}$$

   Setting $\pi' = U' = 0$ and solving, we get the particular integrals $\overline{\pi} = \mu$ and $\overline{U} = \frac{1}{12} - \frac{\mu}{3}$. Since the reduced equation (19.30) now becomes

$$\begin{bmatrix} r + \frac{1}{6} & \frac{1}{2} \\ -\frac{1}{6} & r+1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   the characteristic equation is $r^2 + \frac{7}{6}r + \frac{1}{4} = 0$, with distinct real roots

$$r_1, r_2 = \frac{-7 \pm \sqrt{13}}{12}$$

   . Using these values successively in the above matrix equation, we find

$$\frac{5 - \sqrt{13}}{6}m_1 = n_1$$

and

$$\frac{5+\sqrt{13}}{6}m_2 = n_2$$

Thus the complementarity functions are

$$\begin{bmatrix} \pi_c \\ U_c \end{bmatrix} = \begin{bmatrix} A_1 \\ A_1(\frac{5-\sqrt(13)}{6}) \end{bmatrix} e^{\frac{-7+\sqrt{13}}{12}t} + \begin{bmatrix} A_2 \\ A_2(\frac{5+\sqrt(13)}{6}) \end{bmatrix} e^{\frac{-7-\sqrt{13}}{12}t}$$

which, when added to the particular integrals, give the general solutions.

4. (a) With $\alpha - T = \frac{1}{2}$, $\beta = 3$, $g = 1/2$, $j = 1/4$ and $\kappa = 1$, the system (19.36) becomes

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 4 \end{bmatrix} \begin{bmatrix} \pi_{t+1} \\ U_{t+1} \end{bmatrix} + \begin{bmatrix} -\frac{7}{8} & \frac{3}{4} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \pi_t \\ U_t \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{2} - \mu \end{bmatrix}$$

Letting $\overline{\pi} = \pi_t = \pi_{t+1}$ and $\overline{U} = U_t = U_{t+1}$ and solving, we get the particular solutions

$$\overline{\pi} = \mu \qquad and \qquad \overline{U} = \frac{1}{6}(1-\mu)$$

. Since the reduced equation (19.38) now becomes

$$\begin{bmatrix} b - \frac{7}{8} & \frac{3}{4} \\ -\frac{1}{2}b & 4b - 1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the characteristic equation is $4b^2 - \frac{33}{8}b + \frac{7}{8} = 0$, with distinct real roots

$$b_1, b_2 = \frac{33 \pm \sqrt{193}}{64}$$

. Using these values successively in the above matrix equation, we find the proportionality relations

$$\frac{23-\sqrt{193}}{48}m_1 = n_1$$

and

$$\frac{23+\sqrt{193}}{48}m_2 = n_2$$

Thus the complementarity functions are

$$\begin{bmatrix} \pi_c \\ U_c \end{bmatrix} = \begin{bmatrix} A_1 \\ A_1(\frac{23-\sqrt(193)}{48}) \end{bmatrix} (\frac{33+\sqrt(193)}{64})^t + \begin{bmatrix} A_2 \\ A_2(\frac{23+\sqrt(193)}{48}) \end{bmatrix} (\frac{33-\sqrt(193)}{64})^t$$

which, when added to the particular solutions, give the general solutions.

(b) With $\alpha - T = \frac{1}{4}$, $\beta = 4$, $g = 1$, $j = 1/4$ and $\kappa = 1$, the system (19.36) becomes

$$\begin{bmatrix} 1 & 0 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} \pi_{t+1} \\ U_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \pi_t \\ U_t \end{bmatrix} = \begin{bmatrix} \frac{1}{16} \\ \frac{1}{4} - \mu \end{bmatrix}$$

The particular solutions are $\overline{\pi} = \mu$ and $\overline{U} = \frac{1}{16}$. Since the reduced equation (19.38) now becomes

$$\begin{bmatrix} b-1 & 1 \\ -b & 5b-1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the characteristic equation is $5b^2 - 5b + 1 = 0$, with distinct real roots

$$b_1, b_2 = \frac{5 \pm \sqrt{5}}{10}$$

. Using these values successively in the above matrix equation, we find

$$\frac{5 - \sqrt{5}}{10} m_1 = n_1$$

and

$$\frac{5 + \sqrt{5}}{10} m_2 = n_2$$

Thus the complementarity functions are

$$\begin{bmatrix} \pi_c \\ U_c \end{bmatrix} = \begin{bmatrix} A_1 \\ A_1(\frac{5-\sqrt{5}}{10}) \end{bmatrix} (\frac{5+\sqrt{5}}{10})^t + \begin{bmatrix} A_2 \\ A_2(\frac{5+\sqrt{5}}{10}) \end{bmatrix} (\frac{5-\sqrt{5}}{10})^t$$

which, when added to the particular solutions, give the general solutions.

Exercise 19.5

1. By introducing a new variable $x \equiv y'$ (which implies that $x' \equiv y''$), the given equation can be rewritten as the system

$$x' = f(x, y)$$

$$y' = x$$

which constitutes a special case of (19.40).

2. Since $\frac{\partial x'}{\partial y} = f_y > 0$, as y increases (moving northware in the phase space), x' will increase (x' will pass through three stages in its sign, in the order: $-, 0, +$). This yields the same conclusion as $\frac{\partial x'}{\partial x}$. Similarly, $\frac{\partial y'}{\partial x} = g_x > 0$ yields the same conclusion as $\frac{\partial y'}{\partial y}$.

3. N/A

4. (a) The $x' = 0$ curve has zero slope, and the $y' = 0$ curve has infinite slope. The equilibrium is a saddle point.

   (b) The equilibrium is also a saddle point.

5. (a) The partial-derivative signs imply that the $x' = 0$ curve is positively sloped, and the $y' = 0$ curve is negatively sloped.

(b) A stable node results when a steep $x' = 0$ curve is coupled with a flat $y' = 0$ curve. A stable focus results if a flat $x' = 0$ curve is coupled with a steep $y' = 0$ curve.

Exercise 19.6

1. (a) The system has a unique equilibrium $E = (0, 0)$. The Jacobian evaluated at E is

$$J_E = \begin{bmatrix} e^x & 0 \\ ye^x & e^x \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $|J_E| = 1$ and $tr(J_E) = 2$, E is locally an unstable node.

(b) There are two equilibriums: $E_1 = (0, 0)$ and $E_2 = (\frac{1}{2}, -\frac{1}{4})$. The Jacobian evaluated at E_1 and E_2 yields

$$J_{E1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

and

$$J_{E2} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

Since $|J_{E1}| = 1$ and $tr(J_{E1}) = 2$, E_1 is locally an unstable node. The second matrix has a negative determinant, thus E_2 is locally a saddle point.

(c) A single equilibrium exists at $(0, 0)$. And

$$J_E = \begin{bmatrix} 0 & -e^y \\ 5 & -1 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 5 & -1 \end{bmatrix}$$

Since $|J_E| = 5$ and $tr(J_E) = -1$, E is locally an stable focus.

(d) A single equilibrium exists at $(0, 0)$. And

$$J_E = \begin{bmatrix} 3x^2 + 6xy & 3x^2 + 1 \\ 1 + y^2 & 2xy \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since $|J_E| = -1$, E is locally a saddle point.

2. (a) The elements of Jacobian are signed as follows: $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$. Thus its determinant is negative, implying that the equilibrium is locally a saddle point.

(b) The elements of Jacobian are signed as follows: $\begin{bmatrix} 0 & - \\ - & 0 \end{bmatrix}$. Thus its determinant is negative, implying that the equilibrium is locally a saddle point.

(c) The elements of Jacobian are signed as follows: $\begin{bmatrix} - & + \\ - & - \end{bmatrix}$. Thus its determinant is positive and its trace negative, implying that the equilibrium is locally either a stable focus or a stable node.

3. The differential equations are

$$p' = h(1 - \mu)$$

$$\mu' = \mu(p + q - m(p))$$

, where $m'(p) < 0$. The equilibrium E occurs where $\overline{p} = p_1$ (where $p_1 = m(p_1) - q$ is the value of p that satisfies (19.56)) and $\overline{\mu} = 1$. The Jacobian is

$$J_E = \begin{bmatrix} 0 & -h \\ \mu(1 - m'(p)) & p + q - m(p) \end{bmatrix}_E = \begin{bmatrix} 0 & -h \\ 1 - m'(p_1) & 0 \end{bmatrix}$$

Since $|J_E| = h(1 - m'(p_1)) > 0$ and $tr(J_E) = 0$, E is locally a vortex – the same conclusion as in the phase diagram analysis.

4. (a) The $x' = 0$ and $y' = 0$ curves share the same equation $y = -x$. Thus the two curves coincide, to give rise to a lineful of equilibrium points. Initial points off that line do not lead to equilibrium.

(b) Since $x' = y' = 0$, neither x nor y can move. Thus any initial position can be considered as an equilibrium.

CHAPTER 20

Exercise 20.2

1.

$$\begin{aligned}\lambda^* &= 1-t\\ \mu^* &= (1-t)/2\\ y^* &= \frac{t}{2}-\frac{t^2}{4}+2\end{aligned}$$

2. The hamiltonian is $H = 6y + \lambda y + \lambda u$ (linear in u). Thus to maximize H, we have $u = 2$ (if λ is positive) and $u = 0$ (if λ is negative)

   From $\lambda' = -\partial H/\partial y = -6 - \lambda$, we find that $\lambda(t) = ke^{-t} - 6$, but since $\lambda(4) = 0$ from the transversality condition, we have $k = 6e^4$, and

   $$\lambda^*(t) = 6e^{4-t} - 6$$

   which is positive for all t in the interval [0,4]. Hence the optimal control is $u^*(t) = 2$. From $y' = y + u = y + 2$, we obtain $y(t) = ce^t - 2$. Since $y(0) = 10$, then $c = 12$, and

   $$y^*(t) = 12e^t - 2$$

   The optimal terminal state is

   $$y^*(4) = 12e^4 - 2$$

3. From the maximum principle, the system of differential equations are

   $$\begin{aligned}\lambda' &= -\lambda\\ y' &= y+\frac{a+\lambda}{2b}\end{aligned}$$

   solving first for λ, we get $\lambda(t) = c_0e^{-t}$. Using $\lambda(T) = 0$ yields $c_0 = 0$. Therefore,

   $$u(t) = \frac{-a}{2b}$$

   and

   $$y(t) = \left(y_0 + \frac{a}{2b}\right)e^t - \frac{a}{2b}$$

4. The maximum principle yields $u = (y+\lambda)/2$, and the following system of differential equations

$$\begin{aligned} \lambda' &= -(u-2y) \\ y' &= u \end{aligned}$$

with the boundary conditions $y(0) = y_0$ and $\lambda(T) = 0$. Substituting for u in the system of equations yields

$$\begin{bmatrix} \lambda' \\ y' \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \lambda \\ y \end{bmatrix}$$

The coefficient matrix has a determinant of -1, the roots are ± 1. For $r_1 = 1$, the eigenvector is

$$\begin{bmatrix} -\frac{1}{2} - 1 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} - 1 \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = 0$$

which yields $m = 1$ and $n = 1$.For $r_2 = -1$, the eigenvector is $m = 1$ and $n = -1/3$. The complete solutions are the homogeneous solutions,

$$\begin{bmatrix} \lambda(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} c_1 e^t + \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix} c_2 e^t$$

From the transversality conditions we get

$$\begin{aligned} c_1 &= \frac{x_0 e^{-2T}}{1/3 + e^{-2T}} \\ c_2 &= \frac{-x_0}{1/3 + e^{-2T}} \end{aligned}$$

The final solution is

$$\begin{aligned} \lambda(t) &= \frac{x_0}{1/3 + e^{-2T}} \left(e^{t-2T} - e^{-t} \right) \\ y(t) &= \frac{x_0}{1/3 + e^{-2T}} \left(e^{t-2T} + \frac{1}{3} e^{-t} \right) \\ u(t) &= \frac{x_0}{1/3 + e^{-2T}} \left(e^{t-2T} - \frac{1}{3} e^{-t} \right) \end{aligned}$$

5. N/A

6. $\lambda^* = 3e^{4-t} - 3 \quad \mu^* = 2 \quad y^* = 7e^t - 2$